Dynamics of Markups, Concentration and Product Spans

Elhanan Helpman and Benjamin Niswonger*
Harvard University

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Abstract

Markups and concentration have recently increased. We develop a simple model with a finite number of multi-product firms that populate an industry together with a continuum of single-product firms and study the dynamics of this industry that arises from investments in new products. The model predicts rising markups and concentration and a declining labor share. We then examine the dynamics of market shares and product spans in response to technical change in the technologies of the multi-product and single product firms and the impact of these changes on the steady state distribution of market shares and product spans.

Keywords: single- and multi-product firms, firm dynamics, industry dynamics, markup, market share, product span

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*Helpman: Department of Economics, Harvard University, Cambridge, MA 02138 and NBER (e-mail: ehelpman@harvard.edu); Niswonger: Department of Economics, Harvard University, Cambridge, MA 02138 (e-mail: niswonger@g.harvard.edu).

1 Introduction

A number of recent studies have investigated the evolution of markups and the growth of concentration in U.S. industries, finding that both markups and concentration have increased; see Autor et al. (2020); De Loecker et al. (2020). These studies find that the ascent in average markups was driven by rising markups of the largest firms and market share reallocation from low- to high-markup firms. Contemporaneously, the labor share declined.

We propose a simple model that generates these patterns and study its implications for the dynamism of firms and industries. Our industry has a continuum of varieties of a differentiated product and it is populated by a continuum of single-product firms and a finite number of large multi-product firms. While the turnover of single-product firms is very high, the large firms have long life spans, but they lose product lines and replace them via investment. Free entry of the single-product firms, which engage in monopolistic competition, creates a competitive fringe that influences the oligopolistic competition among the large firms, and the interaction between the single- and multi-product firms plays a key role in the dynamics of the industry, both during transition and in steady state.[1] We show, for example, that in steady state large firms with lower marginal costs have larger market shares, yet some of them may have a lower number of products, i.e., lower product span, in which case the relationship between market share and product span has an inverted U shape. Moreover, in this case an improvement in the technology of single-product firms that raises the competitive pressure on large firms, leads to a decline in the market share of all large firms on impact and to transition dynamics that vary across the large firms according to size. In particular, multi-product firms with large market shares compensate for the initial loss of market share by gradually expanding their product span and market share, while firms with small market shares further reduce their product span and market shares. As a result, the size distribution of multi-product firms becomes more unequal in the new steady state.

We describe some basic elements of the model in the next section. In Section 3 we detail the entry decisions of single-product firms and their impact on the pricing strategy of large firms. These results are then used in Section 4 to study the investment decisions of large firms and the resulting transition dynamics. We show that whenever multi-product firms widen their product span in the transition, they grow in size and so do their markups, while the labor share declines. In the following Section 5 we study comparative dynamics, some of which were described above. Section 6 concludes.

2 Preliminaries

We consider an economy with a continuum of individuals of mass 1, each one providing l units of labor. The labor market is competitive and every individual earns the same wage rate.

There are two sectors. One sector produces a homogeneous good with one unit of labor per unit output and this sector is always active. We normalize the price of this good to equal one and

[1] Interactions between a competitive fringe of single-product firms and oligopolistic large firms have been studied in a static framework by Shimomura and Thisse (2012) in a closed economy and by Parenti (2018) in an open economy.

therefore the competitive wage also equals one. The other sector produces varieties of a differentiated product.[2]

Every individual has a utility function:

$$u = x_0 + \frac{\varepsilon}{\varepsilon - 1}\left[\int_0^N x(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\right]^{\frac{(\varepsilon-1)\sigma}{\varepsilon(\sigma-1)}},\ \sigma > \varepsilon > 1, \tag{1}$$

where x_0 is consumption of the homogeneous good, $x(\omega)$ is consumption of variety ω of the differentiated product, σ is the elasticity of substitution between varieties of the differentiated product and ε gauges the degree of substitutability between varieties of the differentiated product and the homogeneous good. The assumption $\sigma > \varepsilon$ secures that brands of the differentiated product are better substitutes for each other than they are for the homogeneous good. The assumption $\varepsilon > 1$ ensures that aggregate spending on the differentiated product declines as its price rises (see below).

Real consumption of the differentiated product is:

$$X = \left[\int_0^N x(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}}.$$

Using this definition, the price index of X is:

$$P = \left[\int_0^N p(\omega)^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

where $p(\omega)$ is the price of variety ω. And utility maximization under the budget constraint $x_0 + PX = I$, where I is income, yields $X = P^{-\varepsilon}$ as long as consumers purchase the homogenous good as well as X, which we assume is always the case. This arises when l is large enough. The demand for variety ω is:

$$x(\omega) = P^{\delta} p(\omega)^{-\sigma}, \delta = \sigma - \varepsilon > 0. \tag{2}$$

Aggregate spending on the differentiated product equals $PX = P^{1-\varepsilon}$, which declines in P, because $\varepsilon > 1$.

Two types of firms operate in sector X: atomless single-product firms and large multi-product firms, each one with a positive measure of products. Single-product firms produce $\overline{r} > 0$ varieties, each one specializing in a single brand. Large firm i has $r_i > 0$ brands, $i = 1, 2, ..., m$, where m is the number of large firms. All the brands supplied by the single-product and large firms are distinct from each other.

All single-product firms have the same technology, which requires $\bar{a}$ unit of labor per unit output. Facing the demand function (2), a single-product firm maximizes profits $P^{\delta} p(\omega)^{-\sigma}\left[p(\omega) - \bar{a}\right]$, taking as given the price index P. Therefore a single-product firm prices its brand ω according to $p(\omega) = \overline{p}$, where:

$$\overline{p} = \frac{\sigma}{\sigma - 1}\bar{a}. \tag{3}$$

[2] It is straightforward to generalize the analysis to multiple sectors with differentiated products.

This yields the standard markup $\overline{\mu} = \sigma/(\sigma - 1)$ for a monopolistically competitive firm.

A large firm i has a technology that requires a_i units of labor per unit output and it faces the demand function (2) for each one of its brands. As a result, it prices every brand equally. We denote this price by p_i. The firm chooses p_i to maximize profits $r_i P^{\delta} p_i^{-\sigma}(p_i - a_i)$. However, unlike a single-product firm, a large firm does not view P as given, because it recognizes that:

$$P = \left(\overline{r}\,\overline{p}^{1-\sigma} + \sum_{j=1}^{m} r_j p_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}, \tag{4}$$

and therefore its pricing policy has a measurable impact on the price index of the differentiated product. Accounting for this dependence of P on the firm's price, the profit maximizing price is:

$$p_i = \frac{\sigma - \delta s_i}{\sigma - \delta s_i - 1} a_i, \tag{5}$$

where s_i is the market share of firm i and:[3]

$$s_i = \frac{r_i p_i^{1-\sigma}}{P^{1-\sigma}} = \frac{r_i p_i^{1-\sigma}}{\overline{r}\,\overline{p}^{1-\sigma} + \sum_{j=1}^{m} r_j p_j^{1-\sigma}}. \tag{6}$$

Equations (5) and (6) jointly determine prices and market shares of large firms. The markup factor of firm i is $\mu_i = (\sigma - \delta s_i)/(\sigma - \delta s_i - 1)$, which is increasing in its market share. When the market share equals zero the markup is $\sigma/(\sigma - 1)$, the same as the markup of a single product firm. The markup factor varies across firms as a result of differences in either r_i or a_i. We next analyze the dependence of prices, market shares and markups on the firms' marginal costs and product spans.

3 Entry of Single-Product Firms

Consider an economy with given product spans r_i (we discuss dynamics of r_i in the next section). Unlike large firms, single-product firms enter the industry until their profits equal zero. Firms in this sector play a two-stage game: in the first stage single-product firms enter; in the second stage all firms play a Bertrand game as described in the previous section. Under these circumstances, (3) and (5) portray the equilibrium prices, except that the number of single product firms, $\bar{r}$, is endogenous. We seek to characterize a subgame perfect equilibrium of this game.

To determine the equilibrium number of single-product firms, assume that they face an entry cost f and they enter until profits equal zero. In a subgame perfect equilibrium every entrant correctly forecasts the number of entrants, and the price that will be charged for every variety in the second stage of the game. Therefore, every single-product firm correctly forecasts the price index P. Using the optimal price (3) and the profit function $P^{\delta}\overline{p}^{-\sigma}(\overline{p} - \bar{a})$, this free entry condition can

[3] Note that $\sigma - \delta s_i - 1 = \sigma(1 - s_i) + \varepsilon s_i - 1 > 0$ and $\sigma - \delta s_i = \sigma(1 - s_i) + \varepsilon s_i > 0$.

be expressed as:

$$\frac{1}{\sigma}P^{\delta}\left(\frac{\sigma}{\sigma-1}\bar{a}\right)^{1-\sigma}=f. \tag{7}$$

The left-hand side of this equation describes the operating profits, which equal a fraction $1/\sigma$ of revenue, while the right-hand side represents the entry cost. In these circumstances the price index P is determined by f and $\bar{a}$, and it is rising in both f and $\bar{a}$. Importantly, it does not depend on the number of large firms nor on their product range.

We now use (5) and (6) to calculate the response of prices and market shares to changes in the range of products, changes in marginal costs, and changes in the price index P. Denoting by a hat the proportional rate of change of a variable, i.e., $\hat{x}=dx/x$, differentiating these two equations yields the solutions:

$$\hat{p}_i=\frac{\beta_i}{1+(\sigma-1)\beta_i}\hat{r}_i+\frac{1}{1+(\sigma-1)\beta_i}\hat{a}_i+\frac{(\sigma-1)\beta_i}{1+(\sigma-1)\beta_i}\hat{P}, \tag{8}$$

$$\hat{s}_i=\frac{1}{1+(\sigma-1)\beta_i}\hat{r}_i-\frac{\sigma-1}{1+(\sigma-1)\beta_i}\hat{a}_i+\frac{\sigma-1}{1+(\sigma-1)\beta_i}\hat{P}. \tag{9}$$

where:

$$\beta_i=\frac{\delta s_i}{(\sigma-\delta s_i-1)(\sigma-\delta s_i)}>0. \tag{10}$$

Due to the fact that the price index P responds neither to changes in r_i nor changes in a_i, an increase in r_i raises p_i and s_i, but it has no impact on prices and market shares of the other large firms. For the same reason an increase in a_i raises p_i and reduces s_i, but has no impact on prices and market shares of the other large firms. Moreover, an increase in a_i raises the price of firm i less than proportionately, and therefore there is only partial pass-through of marginal costs to prices. The extent of the pass-through is smaller for a firm with a larger β_i, which is a firm with a larger market share. Finally, an increase in the price index P, which represents a decline in the competitive pressure, raises the price and the market share of every large firm. However, the price rises proportionately more and the market share rises proportionately less in firms with larger β_is, which are firm with larger market shares. And the market share of a firm is larger the larger is its product span or the lower is its marginal cost of production. Noting again that the markup of every firm i is larger the larger its market share, we summarize these findings in

Proposition 1. *Suppose that the number of large firms and their product spans are given, but there is free entry of single-product firms. Then: (i) an increase in r_i raises the price, markup and market share of firm i, but has no impact on prices, markups and market shares of other large firms; (ii) a decline in a_i reduces the price and raises the markup and market share of firm i, but has no impact on prices, markups and market shares of other large firms; (iii) a decline in the price index P, either due to a decline in $\bar{a}$ or a decline in f, reduces the price, markup and market share of every large firm, with prices changing proportionately more and market shares changing proportionately less for* *firms with initially larger market shares.*

It is clear from this proposition that free entry of single-product firms leads large firms to compete for market share with single-product firms rather than with each other. An increase in r_i or a decline in a_i, which raise the market share of firm i, do not impact the market share of other large firms, but it reduces the market share of single-product firms. Since the price index P does not change in response to changes in r_i or a_i, the decline in the market share of the single-product firms is attained via a decline in $\bar{r}$. We therefore have

Proposition 2. *Suppose that the number of large firms and their product spans are given, but there is free entry of single-product firms. Then a decline in a_i or an increase in r_i reduces the number of single-product firms and their market share.*

4 Transition Dynamics

We next consider the dynamics that arise when large firms can expand their product range. Time is continuous and the economy starts at time $t = 0$. The range of products of firm i at time t is $r_i(t)$ for $t \geq 0$ and $r_i(0) = r_i^0$ is given.

At every point in time firm i can invest in order to increase its product range. An investment flow of ι_i per unit time generates a cost ι_i per unit time and expands r_i by $\phi(\iota_i)$ per unit time, where $\phi(\iota_i)$ is an increasing and concave function and $\phi(0) = 0$. Furthermore, r_i depreciates at the rate θ per unit time, which randomly hits the continuum of available brands. It follows that r_i satisfies the differential equation:

$$\dot{r}_i = \phi(\iota_i) - \theta r_i, \text{ for all } t \geq 0, \tag{11}$$

where we have suppressed the time index t in $\dot{r}_i$, r_i and ι_i.

At every point in time the firms play a two stage game. In the first stage single-product firms enter and large firms invest in their brands. Single-product firms live only one instant of time. For this reason they make profits only in this single instant. This assumption captures in extreme form the empirical property that the turnover of plants of small firms is much larger than the turnover of plants of large firms. In the second stage all firms choose prices, in the manner described in the previous section. Under the circumstances the price index P is determined by the free entry condition (7), and it remains constant as long as the cost of entry and the cost of production of the single-product firms do no change. It follows that the profit flow of large firm i is:

$$\pi_i = r_i P^\delta p_i^{-\sigma}(p_i - a_i) - \iota_i, \text{ for all } t \geq 0,$$

where P is the same at every t while π_i, r_i, p_i and ι_i change over time and p_i is given by (5).

In this economy the state vector is $\mathbf{r} = (r_1, r_2, ..., r_m)$, a function of time t, and the price p_i is a function of $\mathbf{r}$. Note, however, from (8) and (9) that p_i and s_i depend only on the element r_i of $\mathbf{r}$.

We therefore can express the profit function as:

$$\pi_i\left(\iota_i, r_i\right) = r_i P^{\delta} p_i\left(r_i\right)^{-\sigma}\left[p_i\left(r_i\right) - a_i\right] - \iota_i, \text{ for all } t \geq 0, \tag{12}$$

where $p_i\left(r_i\right)$ is the price of firm i's brands as a function of r_i. This firm's market share is also a function of r_i, $s_i\left(r_i\right)$. From (8) and (9) we obtain the elasticities of the functions $p_i\left(r_i\right)$ and $s_i\left(r_i\right)$:

$$\frac{\partial p_i}{\partial r_i}\frac{r_i}{p_i} = \frac{\beta_i}{1+(\sigma-1)\beta_i}, \tag{13}$$

$$\frac{\partial s_i}{\partial r_i}\frac{r_i}{s_i} = \frac{1}{1+(\sigma-1)\beta_i}, \tag{14}$$

where β_i is defined in (10). Note that β_i is increasing in s_i, and that due to (9), s_i is increasing in r_i. Therefore β_i is increasing in r_i. It follows that the elasticity of the price function is larger the larger is s_i while the elasticity of the market share function is smaller the larger is s_i.

Next assume that the interest rate is constant and equal to ρ. This interest rate can be derived from the assumption that individuals discount future utility flows (1) with a constant rate ρ, so that they maximize the discounted present value of utility $\int_0^\infty e^{-\rho t}u(t)dt$. Under these circumstances firm i maximizes the discounted present value of its profits net of investment costs. It therefore solves the following optimal control problem:

$$\max_{\{\iota_i(t), r_i(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\pi_i\left[\iota_i\left(t\right), r_i\left(t\right)\right]dt$$

subject to (11), (12), $r_i(0) = r_i^0$, and a transversality condition to be described below. In this problem ι_i is a control variable while r_i is a state variable. The current value Hamiltonian of this problem is:

$$\mathcal{H}(\iota_i, r_i, \lambda_i) = \left\{r_i P^{\delta} p_i\left(r_i\right)^{-\sigma}\left[p_i\left(r_i\right) - a_i\right] - \iota_i\right\} + \lambda_i\left[\phi\left(\iota_i\right) - \theta r_i\right],$$

where λ_i is the co-state variable of constraint (11). The co-state variable λ_i varies over time. The first-order conditions of this optimal control problem are:

$$\frac{\partial \mathcal{H}}{\partial \iota_i} = -1 + \lambda_i \phi'\left(\iota_i\right) = 0,$$

$$-\frac{\partial \mathcal{H}}{\partial r_i} = -\frac{\partial\left[r_i P^{\delta} p_i^{-\sigma}(p_i - a_i)\right]}{\partial r_i} + \theta\lambda_i = \dot{\lambda}_i - \rho\lambda_i,$$

and the transversality condition is:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_i\left(t\right)r_i(t) = 0.$$

In addition, the optimal path of (ι_i, r_i) has to satisfy the differential equation (11).

The above first-order conditions can be expressed as:

$$\lambda_i \phi'(\iota_i) = 1, \tag{15}$$

$$\dot{\lambda}_i = (\rho + \theta)\lambda_i - P^{\delta} p_i(r_i)^{-\sigma}\left\{p_i(r_i) - a_i - r_i\left(\sigma p_i(r_i)^{-1}[p_i(r_i) - a_i] - 1\right)p'(r_i)\right\}. \tag{16}$$

From (15) we obtain the investment level ι_i as an increasing function of λ_i, which we express as $\iota_i(\lambda_i)$. Substituting this function into (11) yields the autonomous differential equation:

$$\dot{r}_i = \phi[\iota_i(\lambda_i)] - \theta r_i. \tag{17}$$

Next we substitute (5), (10) and (13) into (16) to obtain a second autonomous differential equation:

$$\dot{\lambda}_i = (\rho + \theta)\lambda_i - \Gamma_i(r_i), \tag{18}$$

where:

$$\Gamma_i(r_i) \equiv a_i^{1-\sigma} P^{\delta} \sigma \left[\frac{\sigma - \delta s_i(r_i)}{\sigma - \delta s_i(r_i) - 1}\right]^{-\sigma} \frac{1}{[\sigma - \delta s_i(r_i) - 1]\sigma + s_i(r_i)^2 \delta^2}. \tag{19}$$

The function $\Gamma_i(r_i)$ represents the profitability of a new variety, given the firm's product span r_i; that is, it represents the marginal profitability of r_i. We show in the Appendix that this marginal profitability declines in r_i, i.e., $\Gamma_i'(r_i) < 0$.

A solution to the autonomous system of differential equations (17) and (18) that satisfies the transversality condition is also a solution to the firm's optimal control problem, because $\mathcal{H}(\iota_i, r_i, \lambda_i)$ is concave in the first two arguments. This can be seen by observing that the Hamiltonian is additively separable in ι_i and r_i, and it is strictly concave in ι_i and in r_i. The steady state of these differential equations is characterized by:

$$\phi[\iota_i(\lambda_i)] = \theta r_i, \tag{20}$$

$$(\rho + \theta)\lambda_i = \Gamma_i(r_i). \tag{21}$$

The left-hand side of (20) is an increasing function of λ_i. Therefore the curve in (r_i, λ_i) space along which r_i is constant is upward sloping. The right-hand side of (21) is declining in r_i, because $\Gamma_i'(r_i) < 0$. Therefore the curve in (r_i, λ_i) space along which λ_i is constant is downward sloping. These curves are depicted in Figure 1. Based on the differential equations (17)-(18), the figure also depicts the resulting dynamics. There is a single stable saddle-path along which (r_i, λ_i) converge to the steady state in which the transversality condition is satisfied. On this saddle path either r_i rises and λ_i declines or r_i declines and λ_i rises, depending on whether r_i^0 is below or above its steady state value.

Now suppose that all the r_i^0s are below their steady state values. Then every large firm expands its range of products over time. As a result, the number of single-product firms shrinks. This process continues until the economy reaches a steady state.

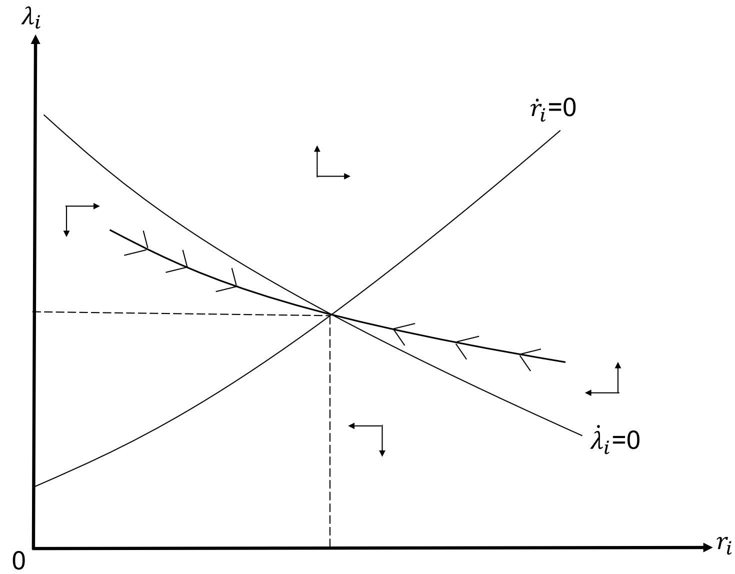

Figure 1: Transition Dynamics

If at some point in time the number of single-product firms drops to zero, the dynamics change.[4] We focus, however, on the case in which $\bar{r} > 0$ for all $t \geq 0$. In this case the price index P remains constant as long as f and $\bar{a}$ do not change.

What can be said about the dynamics of profits of a large firm i? Changes of profits over time can be expressed as:

$$\frac{\partial \pi_i (\iota_i, r_i)}{\partial t} = -\frac{\partial \iota_i}{\partial t} + \frac{\partial \left[r_i P^{\delta} p_i^{-\sigma} (p_i - a_i) \right]}{\partial r_i} \frac{\partial r_i}{\partial t}.$$

From (15) we see that ι_i is an increasing function of λ_i, and since for a firm that expands its product range λ_i declines, it follows that the investment level ι_i also declines over time, raising profits net of investment costs. From (16) and (18) we see that:

$$\frac{\partial \left[r_i P^{\delta} p_i^{-\sigma} (p_i - a_i) \right]}{\partial r_i} = \Gamma_i (r_i) > 0.$$

Therefore, profits grow in a firm that expands r_i over time. We conclude that profits net of investment costs rise in growing firms. Since wages are constant, this implies that in an economy in which all large firms grow, the share of labor in national income declines.

While the share of labor declines over time, average markups rise over time. To see the sources of variation in average markups, note that the average markup μ_{av} can be expressed as a weighted

[4] From that point on the optimal strategy of large firm i depends on the entire state vector $\mathbf{r}$. As a result, the firms engage in a differential game. Since no firm can commit to the entire path of its investments ι_i, one needs to adopt the closed loop solution to this game, in which the investment level ι_i is a function of the state vector $\mathbf{r}$. There do not exist user-friendly characterizations of solutions to such games. Instead, we provide in the Appendix an analysis of the impact of changes in the state variables r_i on prices, markups and market shares of the large firms.

average of the markups of all single- and multi-product firms:

$$\mu_{av} = \left(1 - \sum_{i=1}^{m} s_i\right)\bar{\mu} + \sum_{i=1}^{m} s_i \mu_i,$$

where the markup of a single product firm is $\bar{\mu} = \sigma / (\sigma - 1)$ and the markup of firm i is $\mu_i = (\sigma - \delta s_i) / (\sigma - \delta s_i - 1)$. Since the markup of every large firm is higher than the markup of every single-product firm and the market share of every large firm rises over time, the average markup also increases. The increase in the average markup is driven by two forces: rising markups of the large firms and market share reallocation from low-markup (single-product) to high-markup (multi-product) firms. We summarize these findings in

Proposition 3. *Consider an economy in which the initial range of products r_i^0 is smaller than its steady state value for every i, and in which $\bar{r} > 0$ at all times. Then over time: (i) every large firm widens its product span and raises its markup and profits net of investment costs; (ii) the average markup rises and the labor share declines.*

Since wages are constant, this proposition implies that the growth of large multi-product firms widens the income gap between individuals who own shares in these firms and individuals who do not. As a result, the labor share declines and income inequality widens between these groups of individuals.

5 Comparative Dynamics

For an economy that is initially in steady state, we study in this section the dynamics that arise in response to changes in the marginal costs of production and the cost of entry of single-product firms. As is evident from (20) and (21), such changes impact the new steady state through the function $\Gamma_i(r_i)$ only. A change that raises $\Gamma_i(r_i)$ shifts upward the $\dot{\lambda}_i = 0$ curve in Figure 1. After the impact effect, which results from the upward jump in λ_i, the dynamic process leads to a gradual widening of the span of products and increases in the markup and profits net of investment costs. In contrast, a change that reduces $\Gamma_i(r_i)$ shifts downward the $\dot{\lambda}_i = 0$ curve in Figure 1. After the impact effect, which results from the downward jump in λ_i, the dynamic process leads to a gradual narrowing of the span of products and declines in markups and profits net of investment costs.

First, consider a decline in a_i, resulting from a technical improvement in the firm's technology. We show in the Appendix that the impact of a_i on $\mathit{\Gamma}_i$ can be expressed as:

$$\begin{aligned}\hat{\mathit{\Gamma}}_i &= -(\sigma - 1)\hat{a}_i + \left(\frac{\partial \mathit{\Gamma}_i}{\partial s_i}\frac{s_i}{\mathit{\Gamma}_i}\right)\left(\frac{\partial s_i}{\partial a_i}\frac{a_i}{s_i}\right)\hat{a}_i \\ &= \frac{(\sigma - 1)s_i^2\delta^2 - (\sigma - \delta s_i - 1)^2\left(\sigma^2 - \delta^2 s_i^2\right)}{\left[(\sigma - \delta s_i - 1)\sigma + s_i^2\delta^2\right]^2}(\sigma - 1)\hat{a}_i.\end{aligned} \tag{22}$$

The relationship between a_i and $\mathit{\Gamma}_i$ portrayed by this equation does not depend on the cost structure of other firms. Moreover, it implies that a decline in a_i shifts upward the $\dot{\lambda}_i = 0$ curve if and only if:

$$\left(\sigma - \delta s_i - 1\right)^2 \left(\sigma^2 - \delta^2 s_i^2\right) > \left(\sigma - 1\right) s_i^2 \delta^2. \tag{23}$$

The potential ambiguity of the response of Γ_i to changes in a_i results from the existence of two channels through which the marginal cost impacts the profitability of a new variety (the marginal profitability of r_i), as can be seen from (19). A decline in a_i raises Γ_i for a given market share s_i, due to cost savings in production. But, as shown in (9), a decline in a_i raises the market share of firm i and a rise in the firm's market share reduces the profitability of a new variety. It follows that the shift of the $\dot{\lambda}_i = 0$ curve depends on the strength of these two effects: if the response of the market share dominates, the curve shifts down; and if the response of the market share does not dominate, the curve shifts up. The strength of the market share effect depends in turn on the firm's initial size. For low values of s_i the impact through the market share channel is small, and (23) is satisfied. But (23) is less likely to be satisfied the larger s_i is, because the left-hand side of this inequality is declining in s_i while the right-hand side is increasing. This leads to the following

Lemma 1. *If $\left(\sigma - \delta - 1\right)^2 \left(\sigma^2 - \delta^2\right) > \left(\sigma - 1\right) \delta^2$, then (23) is satisfied for all market shares $s_i \epsilon \left[0, 1\right]$. And if $\left(\sigma - \delta - 1\right)^2 \left(\sigma^2 - \delta^2\right) < \left(\sigma - 1\right) \delta^2$, then there exists a market share $s^o \epsilon \left(0, 1\right)$, defined by:*

$$\left(\sigma - \delta s^o - 1\right)^2 \left[\sigma^2 - \delta^2 \left(s^o\right)^2\right] = \left(\sigma - 1\right) \left(s^o\right)^2 \delta^2,$$

such that (23) is satisfied for $s_i < s^o$ and violated for $s_i > s^o$.

Given the assumption $\sigma > \varepsilon > 1$, the inequality $\left(\sigma - \delta - 1\right)^2 \left(\sigma^2 - \delta^2\right) > \left(\sigma - 1\right) \delta^2$ is satisfied when ε is close to σ and violated when ε is close to one (recall that $\delta = \sigma - \varepsilon$). We therefore have

Proposition 4. *Suppose that firm i is in steady state and $\bar{r} > 0$ at all times. Then a decline in a_i triggers an adjustment process that gradually raises r_i as well as i's markup and profits net of investment costs if either $\left(\sigma - \delta - 1\right)^2 \left(\sigma^2 - \delta^2\right) > \left(\sigma - 1\right) \delta^2$ or $\left(\sigma - \delta - 1\right)^2 \left(\sigma^2 - \delta^2\right) < \left(\sigma - 1\right) \delta^2$ and $s_i < s^o$, where s^o is defined in Lemma 1. Otherwise, this technical improvement triggers an adjustment process that gradually reduces r_i while i's markup and profits net of investment costs decline gradually after increasing on impact.*

Using these results, we can examine the dynamics of firm i's market share. Since on impact the span of products does not change (r_i is a state variable), (9) implies that the decline in the marginal cost raises on impact firm i's market share. Moreover, if the adjustment process leads to a gradual expansion of its product span, i's market share rises over time until it reaches a new steady state. In this case the firm has a larger market share in the new steady state. If, however, the adjustment process leads to a narrowing of the firm's product span, then (9) implies that the initial upward jump in firm i's market share is followed by a gradual decline in its market share. The question then arises whether this firm's market share is larger or smaller in the new steady state. We prove the following

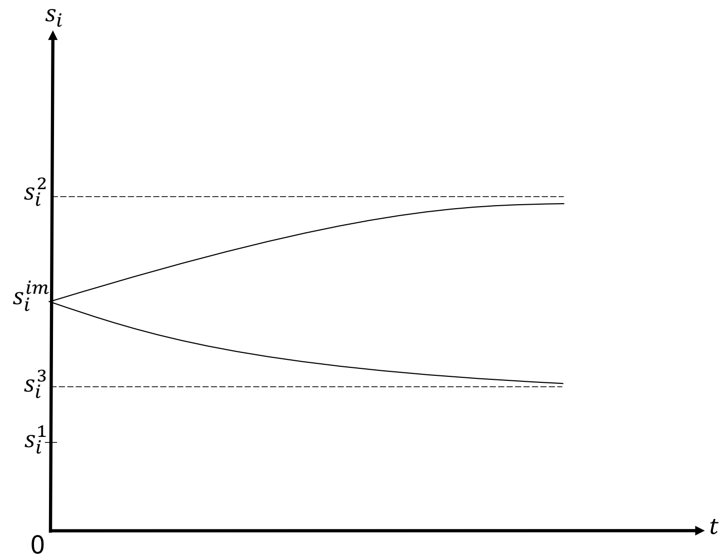

Figure 2: Dynamics of the market share in response to a decline in the marginal cost a_i

Proposition 5. *Suppose that firm i is in steady state and $\bar{r} > 0$ at all times. Then a decline in a_i triggers an adjustment process that raises s_i in the new steady state.*

Proof. We have shown that the market share is larger in the new steady state when the adjustment process involves expansion of the firm's product span. It therefore remains to show that this is also true when the adjustment process involves contraction of the product span. To this end note that a decline in r_i on the transition path is triggered by a decline in the marginal profitability of r_i in response to a decline in a_i, which leads in turn to a downward shift in the $\dot{\lambda}_i = 0$ curve in Figure 1. In this case the new steady state has a lower r_i as well as a lower λ_i. Next note from the steady state condition (21) that a lower λ_i implies a lower Γ_i. Recall, however, that for a constant s_i a fall in a_i raises Γ_i, and therefore Γ_i can be lower in the new steady state only if s_i is higher. In sum, independently of whether a decline a_i shifts upward or downward the $\dot{\lambda}_i = 0$ curve, the market share s_i is larger in the new steady state. □

This result yields the following

Corollary. *Consider an economy in steady state with active single-product firms. Then large firms with lower marginal costs have larger market shares.*

The dynamic patterns of the market share that have been revealed by this analysis are depicted in Figure 2, where s_i^1 is the market share in the initial steady state. First, the market share jumps up to s_i^{im} on impact when a_i declines. Afterward, the market share rises continuously until it reaches s_i^2, as portrayed by the upper curve, or it declines continuously until it reaches s_i^3, as portrayed by the lower curve. In both cases the new steady state market share exceeds s_i^1. The former case applies when $(\sigma - \delta - 1)^2 \left(\sigma^2 - \delta^2\right) > (\sigma - 1)\,\delta^2$ or $(\sigma - \delta - 1)^2 \left(\sigma^2 - \delta^2\right) < (\sigma - 1)\,\delta^2$ and $s_i^1 < s^o$, and the latter case applies otherwise.

These results suggest three possible steady state patterns for the relationship between a_i and r_i in the cross section of multi-product firms: lower-cost firms have larger product spans, lower-cost firms have smaller product spans, or the relationship between marginal costs and product spans has an inverted U shape. The first pattern holds for all marginal cost structures when $(\sigma-\delta-1)^2\left(\sigma^2-\delta^2\right) > (\sigma-1)\,\delta^2$. In the opposite case, when $(\sigma-\delta-1)^2\left(\sigma^2-\delta^2\right) < (\sigma-1)\,\delta^2$, there exist high values of a_i at which $s_i < s^o$, and among firms with such high marginal costs firms with lower marginal costs have larger product spans. Moreover, there exist low values of a_i at which $s_i > s^o$, and among firms with such low marginal costs lower-cost firms have smaller product spans.

Combining these results we have

Proposition 6. *Consider an economy in steady state with active single-product firms. Then, in the cross section of multi-product firms r_i is declining in s_i, rising in s_i, or rising in s_i among firms with low market shares and declining in s_i among firms with high market shares.*

We next examine the impact of the cost structure of single-product firms. As is evident from (7), a decline in either the marginal cost or the entry cost of single-product firms reduces the price index P, thereby raising the competitive pressure in the economy. How do the large firms respond to this rise in competition? To answer the question, suppose that all firms are in steady state. Equation (19) implies:

$$\hat{\Gamma}_i = \delta \hat{P}_i + \left(\frac{\partial \Gamma_i}{\partial s_i}\frac{s_i}{\Gamma_i}\right)\left(\frac{\partial s_i}{\partial P}\frac{P}{s_i}\right)\hat{P}. \tag{24}$$

A decline in the price index P, which elevates the competitive pressure on every large firm, reduces the marginal value of r_i. For this reason the first term on the right-hand side of this equation is negative when $\hat{P}_i < 0$. In response, firm i reduces its price and market share (see (8) and (9)) and the fall in market share raises the marginal value of r_i. For this reason the second term on the right-hand side is positive when $\hat{P}_i < 0$. It follows that a decline in P shifts the $\dot{\lambda}_i = 0$ curve downward in Figure 1 if the competition effect dominates and upward if the market share effect dominates. Using (9), it is evident that for $\varepsilon \to 1$ (24) is similar to (22), except for the opposite sign on their right-hand sides. Therefore, in this case a decline in P shifts down the $\dot{\lambda}_i = 0$ curve if and only if a decline in a_i shifts it up. Under these conditions a lower P may lead to a lower or higher value of r_i in steady state, and moreover, its impact may vary across firms with different marginal costs and therefore different market shares s_i. For $\varepsilon \to 1$ the inequality $(\sigma-\delta-1)^2\left(\sigma^2-\delta^2\right) > (\sigma-1)\,\delta^2$ is violated, implying that there exists an s_P^o such that the decline in P shifts the $\dot{\lambda}_i = 0$ curve down for $s_i < s_P^o$ and up for $s_i > s_P^o$. Therefore, in this case a rise in the competitive pressure shrinks the product span of multi-product firms with $s_i < s_P^o$ and expands the product span of multi-product firms with $s_i > s_P^o$. As a result, the gaps in market shares between large and small multi-product firms widens, thereby increasing the inequality in the size distribution of firms.[5] Alternatively, for $\varepsilon \to \sigma$, the competition effect is negligible and the shift in the market share dominates the impact on Γ_i. As a result, the $\dot{\lambda}_i = 0$ curve shifts up for all multi-product firms, raising their product spans.

[5] From (9), $\hat{s}_i - \hat{s}_j = (\hat{r}_i - \hat{r}_j)\,/\,[1 + (\sigma-1)\,\beta_i]$. Therefore $\hat{s}_i > \hat{s}_j$ if and only if $\hat{r}_i > \hat{r}_j$.

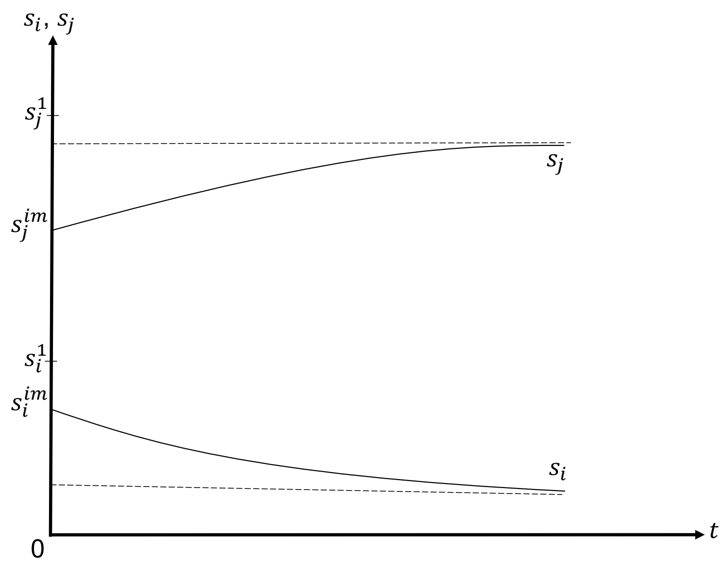

Figure 3: Dynamics of market shares in response to a decline in P

Finally, note that a decline in P reduces the steady state market share of every large firm. This is clearly the case when every firm's product span declines, because in this case both P and r_i diminish the market share (see (9)). Alternatively, for a firm that expands its steady state r_i, the value of λ_i is higher in the new steady state (see (20)). Therefore this firm's Γ_i is also larger in the new steady state (see (21)). But the direct impact of the decline in P on Γ_i is negative, and therefore s_i has to be smaller for Γ_i to be larger. We therefore have

Proposition 7. *Consider an economy in steady state with $\bar{r} > 0$. Then, a technical improvement that reduces either f or $\bar{a}$ may raise r_i in the new steady state for all i, reduce r_i for all i, or reduce r_i of the small multi-product firms and raise r_i of the large multi-product firms. Nevertheless, s_i is smaller in the new steady state for all i.*

Figure 3 depicts the dynamics of two firms, i and j, for the case in which $s_i < s_P^o$ and $s_j > s_P^o$, where s_P^o is the cutoff market share for the opposite firm dynamics. Firm i starts with $s_i = s_i^1$ while firm j starts with $s_j = s_j^1$. In both firms the market share jumps down on impact as a result of the decline in P, to s_i^{im} and s_j^{im}, respectively. After that the market share of the smaller firm declines while the market share of the larger firm rises. Yet in both cases the market share is lower in the new steady state.

6 Conclusion

We have developed a simple model of firm dynamics that generates time patterns of markups, concentration and labor shares that are consistent with the data. Our model features distinct roles for single- and multi-product firms. Investment of multi-product firms in new varieties plays a key role in these dynamics. The model predicts interesting relationships between market shares and product spans during transition dynamics and in steady state. There are few data sets containing information on product span of individual firms, and these data are confidential. We nevertheless hope that our predictions will eventually be tested.

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Appendix

Comparative Dynamics

We first derive the slope of the $\dot{\lambda}_i$=0 curve. Differentiation of the right-hand side of (21) yields:

$$\hat{\Gamma}_i = -(\sigma-1)\hat{a}_i + \delta\hat{P} - \frac{\sigma\delta s_i}{(\sigma-\delta s_i-1)(\sigma-\delta s_i)}\hat{s}_i + \frac{\delta s_i(\sigma-2\delta s_i)}{(\sigma-\delta s_i-1)\sigma+\delta^2 s_i^2}\hat{s}_i.$$

This equation implies that the right-hand side of (21) is declining in r_i because Γ_i is declining in s_i and s_i is rising in r_i (see (9)). The former is seen from this equation by observing that $\sigma\delta s_i > \delta s_i(\sigma-2\delta s_i)$ and $(\sigma-\delta s_i-1)(\sigma-\delta s_i)<(\sigma-\delta s_i-1)\sigma+s_i^2\delta^2$. Collecting terms we can rewrite this equation as:

$$\hat{\Gamma}_i = -(\sigma-1)\hat{a}_i + \delta\hat{P} - \delta^2 s_i^2 \frac{2(\sigma-\delta s_i-1)(\sigma-\delta s_i)+\sigma(\sigma-1)}{(\sigma-\delta s_i-1)(\sigma-\delta s_i)\left[(\sigma-\delta s_i-1)\sigma+\delta^2 s_i^2\right]}\hat{s}_i. \tag{25}$$

Next consider the total effect of a shift in the marginal cost a_i on Γ_i. From (9) we have:

$$\hat{s}_i = -\frac{\sigma-1}{1+(\sigma-1)\beta_i}\hat{a}_i = -\frac{(\sigma-1)(\sigma-\delta s_i-1)(\sigma-\delta s_i)}{(\sigma-\delta s_i-1)(\sigma-\delta s_i)+(\sigma-1)\delta s_i}\hat{a}_i.$$

Substituting this expression into (25) we obtain the total impact of a_i on Γ_i:

$$\begin{aligned}\frac{\hat{\Gamma}_i}{(\sigma-1)\hat{a}_i} &= -1+\delta^2 s_i^2\frac{2(\sigma-\delta s_i-1)(\sigma-\delta s_i)+\sigma(\sigma-1)}{\left[(\sigma-\delta s_i-1)\sigma+s_i^2\delta^2\right]^2} \\ &= \frac{(\sigma-1)s_i^2\delta^2-(\sigma-\delta s_i-1)^2\left(\sigma^2-\delta^2 s_i^2\right)}{\left[(\sigma-\delta s_i-1)\sigma+s_i^2\delta^2\right]^2}.\end{aligned}$$

It follows that a decline in the marginal cost a_i shifts upward the $\dot{\lambda}_i$=0 curve if and only if $(\sigma-1)s_i^2\delta^2 < (\sigma-\delta s_i-1)^2\left(\sigma^2-\delta^2 s_i^2\right)$.

Comparative Statics: Given Number of Brands

In this section we examine the case in which the number of single-product firms, $\overline{r}$, as well the number of products available to each one of the large firms, r_i, are given. Equations (5) and (6) imply:

$$\hat{p}_i = \hat{a}_i + \frac{\delta s_i}{(\sigma-\delta s_i-1)(\sigma-\delta s_i)}\hat{s}_i, \tag{26}$$

$$\hat{s}_i = \hat{r}_i - \sum_{j=1}^{m} s_j\hat{r}_j - (\sigma-1)(\hat{p}_i - \sum_{j=1}^{m} s_j\hat{p}_j).$$

Substituting the last equation into (26) yields:

$$[1+\beta_i(\sigma-1)]\hat{p}_i - \beta_i(\sigma-1)\sum_{j=1}^{m} s_j\hat{p}_j = \hat{a}_i + \beta_i(\hat{r}_i - \sum_{j=1}^{m} s_j\hat{r}_j), \text{ for all } i.$$

These equations can also be expressed as:

$$\mathbf{B}\hat{\mathbf{p}} = \mathbf{R}\hat{\mathbf{r}} + \hat{\mathbf{a}}, \tag{27}$$

where $\mathbf{B}$ is an $m \times m$ matrix with elements:

$$b_{ii} = 1 + \beta_i(\sigma-1)(1-s_i),$$

$$b_{ij} = -\beta_i(\sigma-1)s_j, \text{ for } j \neq i,$$

$\hat{\mathbf{p}}$ is an $m \times 1$ column vector with elements p_i, where a hat represents a proportional rate of change (i.e., $\hat{p}_i = dp_i/p_i$), $\mathbf{R}$ is an $m \times m$ matrix with elements:

$$r_{ii} = \beta_i(1-s_i),$$

$$r_{ij} = -\beta_i s_j, \text{ for } j \neq i,$$

$\hat{\mathbf{r}}$ is an $m \times 1$ column vector with elements $\hat{r}_i$, where a hat represents a proportional rate of change, and $\hat{\mathbf{a}}$ is an $m \times 1$ column vector with elements $\hat{a}_i$, where a hat represents a proportional rate of change.

Since

$$|b_{ii}| - \sum_{j \neq i} |b_{ij}| = 1 + \beta_i(\sigma-1)(1 - \sum_{j=1}^{m} s_j) > 1,$$

$\mathbf{B}$ is a diagonally dominant matrix with positive diagonal and negative off-diagonal elements. It therefore is an M-matrix and its inverse has all positive entries. This inverse, denoted by $\tilde{\mathbf{B}} = \mathbf{B}^{-1}$, is therefore an $m \times m$ matrix with elements $\tilde{b}_{ij} > 0$. Next note that $\mathbf{B}$ can be expressed as:

$$\mathbf{B} = \mathbf{I} + (\sigma-1)\mathbf{R},$$

where $\mathbf{I}$ is the identity matrix. Therefore:

$$\mathbf{B}^{-1}\mathbf{B} = \tilde{\mathbf{B}} + (\sigma-1)\tilde{\mathbf{B}}\mathbf{R} = \mathbf{I}. \tag{28}$$

It follows from this equation that:

$$\tilde{b}_{ii} + (\sigma-1)\sum_{j=1}^{m} \tilde{b}_{ij} r_{ji} = 1,$$

$$\tilde{b}_{ik} + (\sigma - 1)\sum_{j=1}^{m} \tilde{b}_{ij} r_{jk} = 0, \text{ for } k \neq i.$$

Summing these up yields:

$$\sum_{k=1}^{m} \tilde{b}_{ik} + (\sigma - 1)\sum_{j=1}^{m} \tilde{b}_{ij} \sum_{k=1}^{m} r_{jk} = 1, \text{ for all } i. \tag{29}$$

Since:

$$\sum_{k=1}^{m} r_{jk} = \beta_j (1 - \sum_{k=1}^{m} s_k) > 0$$

and $\tilde{b}_{ik} > 0$ for all i and k, it follows from (29) that:

$$0 < \tilde{b}_{ik} < 1 \text{ for all } i \text{ and } k.$$

Equation (28) implies:

$$(\sigma - 1)\tilde{\mathbf{B}}\mathbf{R} = \mathbf{I} - \tilde{\mathbf{B}},$$

and therefore $\tilde{\mathbf{B}}\mathbf{R}$ has positive diagonal elements and negative off-diagonal elements.

Going back to the comparative statics equations (27), we have:

$$\hat{\mathbf{p}} = \tilde{\mathbf{B}}\mathbf{R}\hat{\mathbf{r}} + \tilde{\mathbf{B}}\hat{\mathbf{a}}.$$

It follows from the properties of $\tilde{\mathbf{B}}$ that a decline in a_i reduces every price p_j, but less than proportionately. Equation (26) than implies that all market share $s_j, j \neq i$, decline while the market share s_i rises. And it follows from the properties of $\tilde{\mathbf{B}}\mathbf{R}$ and (26) that an increase in r_i raises the price and market share of firm i and reduces the price and market share of every other firm $j \neq i$. Noting that the markup of every firm i is larger the larger its market share, we therefore have:

Proposition 8. *Suppose that the number of firms and their product range are given. Then: (i) an increase in r_i raises the price, markup and market share of firm i, and reduces the price, markup and market share of every other large firm; (ii) a decline in a_i reduces the price of every large firm less than proportionately, raises the markup and market share of firm i, and reduces the markup and market share of every other large firms.*