

Dynamics of Markups, Concentration and Product Span

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Abstract

We develop a model with a finite number of multi-product firms that populate an industry together with a continuum of single-product firms, and study the dynamics of this industry that arises from investments in the invention of new products. Consistent with the available evidence, the model predicts rising markups and concentration and a declining labor share. We then examine the dynamics of market shares and product spans in response to improvements in the technologies of the multi-product and single-product firms, and the impact of these changes on the steady state distribution of market shares and product spans. Our model predicts the possibility of an inverted-U relationship between labor productivity and product span in the cross-section of firms, for which we provide suggestive evidence. It also predicts that rising entry costs of single-product firms may flatten the relationship between labor productivity and market shares of the large multi-product firms.

Keywords: single- and multi-product firms, firm dynamics, industry dynamics, markup, market share, product span

JEL Classification: L11, L13, L25, D43

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1 Introduction

A number of recent studies have investigated the evolution of markups and the growth of concentration in U.S. industries, finding that both markups and concentration have increased; see Autor *et al.* (2020); De Loecker *et al.* (2020). These studies find that the ascent in average markups was driven by rising markups of the largest firms and market share reallocation from low- to high-markup firms. Contemporaneously, the labor share declined. We develop a model of firm dynamics that generates these patterns, as well as rich predictions about the unfolding of the cross-section of firms. Our theory focuses on the evolution of a sector rather than on long-run growth of the entire economy.

An industry has a continuum of varieties of a differentiated product and it is populated by a continuum of single-product firms and a finite number of large multi-product firms. While the turnover of single-product firms is very high, the large multi-product firms have long life spans. Large firms lose some products over time, but they can invest in innovation in order to replenish or expand the range of their products. Free entry of the single-product firms, which engage in monopolistic competition, creates a competitive fringe that impacts the oligopolistic competition of the large firms. The interaction between the single- and multi-product firms plays a key role in the dynamics of this industry, both during transition and in steady state.¹

Our assumptions capture salient feature of the data. According to Cao *et al.* (2019), 95% of firms in the U.S. economy are single-establishment firms, but their share in employment is only 45%. Furthermore, Kehrig and Vincent (2019) report that during 1972-2007 an average of 72% of the plants in manufacturing belonged to single-plant firms and 28% belonged to multi-plant firms (see their Table A1). At the same time, single-plant firms manufactured 22% of value added compared to 78% of the value added manufactured by multi-plant firms. Both studies point out that firm growth took place mostly through the extensive margin, by opening new plants that often produced new products. In addition Hsieh and Rossi-Hansberg (2020) provide evidence that firm growth through the acquisition of new product lines played an important role in the business strategies of U.S. corporations in three major sectors: services, retail and wholesale. Growth of firms in these sectors was driven by expansions to new locations, i.e., new product lines. Finally, studying the size distribution of firms between 1995 and 2014, Cao *et al.* (2019) conclude that the largest contributors to the increase in the number of establishments per firm were declining costs of innovation and declining exit rates.

The theory developed in this paper predicts that large multi-product firms grow through innovation that expands their product lines, eventually reaching a steady state. In the processes, these firms raise their markups and reduce their labor share. As a result, the aggregate markup rises and the aggregate labor share declines. The aggregate markup rises as a result of rising markups of

¹Interactions between a monopolistically competitive fringe of single-product firms and oligopolistic large firms have been studied in a static framework by Shimomura and Thisse (2012) in a closed economy and by Parenti (2018) in an open economy. Cavenaile *et al.* (2019) develop a model of endogenous growth with quality-ladders, in which there is a fringe of competitive small firms that produce a homogenous good. In addition, there are single-product large firms that produce different varieties of a product. Their model is mostly quantitative, used to study the relationship between innovation and competition.

the multi-product firms and the reallocation of market shares from single- to multi-product firms. The steady state size distribution of firms is driven by heterogeneity of labor productivity, with more productive firms having larger market shares. Nevertheless, this monotonic relationship does not translate into a monotonic relationship between productivity and product span. The reason is that the marginal profitability of investment in innovation is larger when manufacturing costs are lower, but larger market shares reduce the incentives to invent new products. We show this tension can produce an inverted-U relationship between labor productivity and product span in the cross-section of firms. Using the Compustat data for 2018, we provide evidence that supports this prediction. Our model also predicts that improvements in the technology of single-product firms, which raise the competitive pressure on the multi-product oligopolies, lead to a decline in the market share of every large firm on impact. Still, the resulting transition dynamics to a new steady state vary across the large firms according to size. In particular, multi-product firms with large market shares compensate for the initial loss of competitiveness, as reflected in the loss of market share, by gradually expanding their product span and raising their market share over time, while firms with small market shares further reduce their product span and market shares over time. As a result, the size distribution of multi-product firms becomes more unequal in the new steady state.

We describe some basic elements of the model in the next section. In Section 3 we detail the entry decisions of single-product firms and their impact on the pricing strategy of large firms. These results are then used in Section 4 to study the innovation decisions of large firms and the resulting transition dynamics. We show that whenever multi-product firms widen their product span in the transition, they grow in size and so do their markups, while the labor share declines. In the following Section 5 we study comparative dynamics, some of which were described above. Section 6 concludes.

2 Preliminaries

We consider an economy with a continuum of individuals of mass 1, each one providing l units of labor. The labor market is competitive and every individual earns the same wage rate.

There are two sectors. One sector produces a homogeneous good with one unit of labor per unit output and there is always positive demand for its product. We normalize the price of this good to equal one. Therefore the competitive wage also equals one. The other sector produces varieties of a differentiated product.²

Every individual has a utility function:

$$u = x_0 + \frac{\varepsilon}{\varepsilon - 1} \left[\int_0^N x(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{(\varepsilon-1)\sigma}{\varepsilon(\sigma-1)}}, \quad \sigma > \varepsilon > 1, \quad (1)$$

where x_0 is consumption of the homogeneous good, $x(\omega)$ is consumption of variety ω of the differentiated product, σ is the elasticity of substitution between varieties of the differentiated product and ε gauges the degree of substitutability between varieties of the differentiated product and the

²It is straightforward to generalize the analysis to multiple sectors with differentiated products.

homogeneous good. The assumption $\sigma > \varepsilon$ asserts that brands of the differentiated product are better substitutes for each other than for the homogeneous good. The assumption $\varepsilon > 1$ ensures that aggregate spending on the differentiated product declines when its price rises (see below).

Real consumption of the differentiated product is:

$$X = \left[\int_0^N x(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}}.$$

Using this definition, the price index of X is:

$$P = \left[\int_0^N p(\omega)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}},$$

where $p(\omega)$ is the price of variety ω . In this setup utility maximization subject to the budget constraint $x_0 + PX = I$ yields $X = P^{-\varepsilon}$, where I is income, as long as consumers purchase the homogenous good and varieties of the differentiated product, which we assume always to be the case (this requires l to be large enough). The demand for variety ω is:

$$x(\omega) = P^\delta p(\omega)^{-\sigma}, \delta = \sigma - \varepsilon > 0. \quad (2)$$

Aggregate spending on the differentiated product equals $PX = P^{1-\varepsilon}$, which declines in P , because $\varepsilon > 1$. An individual's consumption choice yields the indirect utility function

$$V = I + \frac{1}{\varepsilon - 1} P^{1-\varepsilon},$$

where the second term on the right-hand side represents consumer surplus. Aggregating across all individuals we obtain the aggregate indirect utility function

$$V_A = I_A + \frac{1}{\varepsilon - 1} P^{1-\varepsilon}, \quad (3)$$

where I_A is aggregate income.

Two types of firms operate in sector X : atomless single-product firms and large multi-product firms, each one with a positive measure of product lines (recall the discussion in the introduction of evidence in support of this specification). Single-product firms produce $\bar{r} > 0$ varieties, each one specializing in a single brand. Large firm i has $r_i > 0$ product lines, $i = 1, 2, \dots, m$, where m is the number of large firms. All the brands supplied to the market are distinct from each other.

All single-product firms share the same technology, which requires $\bar{a}$ unit of labor per unit output.³ Facing the demand function (2), a single-product firm maximizes profits $P^\delta p(\omega)^{-\sigma} [p(\omega) - \bar{a}]$, taking as given the price index P . Therefore, a single-product firm prices its brand ω according to

³It is straightforward to allow for heterogeneity of the single-product firms, by assuming that each one of them draws a unit labor requirement from a known distribution. Since this type of heterogeneity plays no essential role in our analysis, we have chosen to work with the simpler formulation.

$p(\omega) = \bar{p}$, where:

$$\bar{p} = \frac{\sigma}{\sigma - 1} \bar{a}. \quad (4)$$

This yields the standard markup $\bar{\mu} = \sigma / (\sigma - 1)$ for a monopolistically competitive firm.

A large firm i has a technology that requires a_i units of labor per unit output, and it faces the demand function (2) for each one of its brands. As a result, it prices every brand equally. We denote this price by p_i . The firm chooses p_i to maximize profits $r_i P^\delta p_i^{-\sigma} (p_i - a_i)$. However, unlike a single-product firm, a large firm does not view P as given, because it recognizes that

$$P = \left(\bar{r} \bar{p}^{1-\sigma} + \sum_{j=1}^m r_j p_j^{1-\sigma} \right)^{\frac{1}{1-\sigma}}, \quad (5)$$

and therefore that its pricing policy has a measurable impact on the price index of the differentiated product. Accounting for this dependence of P on the firm's price, the profit maximizing price is:

$$p_i = \frac{\sigma - \delta s_i}{\sigma - \delta s_i - 1} a_i, \quad (6)$$

where s_i is the market share of firm i and:⁴

$$s_i = \frac{r_i p_i^{1-\sigma}}{P^{1-\sigma}} = \frac{r_i p_i^{1-\sigma}}{\bar{r} \bar{p}^{1-\sigma} + \sum_{j=1}^m r_j p_j^{1-\sigma}}. \quad (7)$$

Equations (6) and (7) jointly determine prices and market shares of large firms. The markup factor of firm i is $\mu_i = (\sigma - \delta s_i) / (\sigma - \delta s_i - 1)$, which is increasing in its market share. When the market share equals zero the markup is $\sigma / (\sigma - 1)$, the same as the markup of a single product firm. The markup factor varies across firms as a result of differences in either the product span, r_i , or the marginal production cost, a_i . We analyze the dependence of prices, market shares and markups on marginal costs and product spans in the next section.

3 Entry of Single-Product Firms

We take as given the number of large firms, and focus the analysis on the evolution of their product spans, r_i , in the next section. Unlike large firms, single-product firms enter the industry until their profits equal zero. Firms in this sector play a two-stage game: in the first stage single-product firms enter; in the second stage all firms play a Bertrand game as described in the previous section. Under these circumstances, (4) and (6) portray the equilibrium prices, except that the number of single product firms, $\bar{r}$, is endogenous. We seek to characterize a subgame perfect equilibrium of this game.

To determine the equilibrium number of single-product firms, assume that they face an entry cost f and they enter until profits equal zero. In a subgame perfect equilibrium every entrant

⁴Note that $\sigma - \delta s_i - 1 = \sigma(1 - s_i) + \varepsilon s_i - 1 > 0$ and $\sigma - \delta s_i = \sigma(1 - s_i) + \varepsilon s_i > 0$.

correctly forecasts the number of entrants, and the price that will be charged for every variety in the second stage of the game. Therefore, every single-product firm correctly forecasts the price index P . Using the optimal price (4) and the profit function $P^\delta \bar{p}^{-\sigma} (\bar{p} - \bar{a})$, this free entry condition can be expressed as:

$$\frac{1}{\sigma} P^\delta \left(\frac{\sigma}{\sigma-1} \bar{a} \right)^{1-\sigma} = f. \quad (8)$$

The left-hand side of this equation describes the operating profits, which equal a fraction $1/\sigma$ of revenue, while the right-hand side represents the entry cost. In these circumstances the price index P is determined by f and $\bar{a}$, and it is rising in both f and $\bar{a}$. Importantly, it does not depend on the number of large firms nor on their product spans.

We now use (6) and (7) to calculate the response of prices and market shares to changes in the number of product lines, changes in marginal costs, and changes in the price index P . Denoting by a hat the proportional rate of change of a variable, i.e., $\hat{x} = dx/x$, differentiating these two equations yields the solutions:

$$\hat{p}_i = \frac{\beta_i}{1 + (\sigma-1)\beta_i} \hat{r}_i + \frac{1}{1 + (\sigma-1)\beta_i} \hat{a}_i + \frac{(\sigma-1)\beta_i}{1 + (\sigma-1)\beta_i} \hat{P}, \quad (9)$$

$$\hat{s}_i = \frac{1}{1 + (\sigma-1)\beta_i} \hat{r}_i - \frac{\sigma-1}{1 + (\sigma-1)\beta_i} \hat{a}_i + \frac{\sigma-1}{1 + (\sigma-1)\beta_i} \hat{P}. \quad (10)$$

where:

$$\beta_i = \frac{\delta s_i}{(\sigma - \delta s_i - 1)(\sigma - \delta s_i)} > 0. \quad (11)$$

Due to the fact that the price index P responds neither to changes in r_i nor changes in a_i , an increase in r_i raises p_i and s_i , but it has no impact on prices and market shares of the other large firms. For the same reason, an increase in a_i raises p_i and reduces s_i , but has no impact on prices and market shares of the other large firms. Moreover, an increase in a_i raises the price of firm i less than proportionately, and therefore there is only partial pass-through of marginal costs to prices. The extent of the pass-through is smaller for a firm with a larger β_i , which is a firm with a larger market share. Finally, an increase in the price index P , which represents a decline in the competitive pressure in the industry, raises the price and the market share of *every* large firm. However, the price rises proportionately more and the market share rises proportionately less in firms with larger β_i s, which are firm with larger market shares. Finally, the market share of a firm is larger the larger is its product span or the lower is its marginal cost of production. Noting again that the markup of every firm i is larger the larger its market share, we summarize these findings in

Proposition 1. *Suppose that the number of large firms and their product spans are given, but there is free entry of single-product firms. Then: (i) an increase in r_i raises the price, markup and market share of firm i , but has no impact on prices, markups and market shares of the other large firms; (ii) a decline in a_i reduces the price and raises the markup and market share of firm i , but has no*

impact on prices, markups and market shares of other large firms; (iii) a decline in the price index P , either due to a decline in $\bar{a}$ or a decline in f , reduces the price, markup and market share of every large firm, with prices changing proportionately more and market shares changing proportionately less for firms with initially larger market shares.

It is clear from this proposition that free entry of single-product firms leads large firms to compete for market share with single-product firms rather than with each other.⁵ An increase in r_i or a decline in a_i , each of which raises the market share of firm i , does not impact the market share of other large firms, but do reduce the market share of single-product firms. Since the price index P does not change in response to changes in the number of product lines r_i or the marginal cost a_i , the decline in the market share of the single-product firms is attained via a decline in their joint product span $\bar{r}$. We therefore have

Proposition 2. *Suppose that the number of large firms and their product spans are given, but there is free entry of single-product firms. Then a decline in a_i or an increase in r_i reduces the number of single-product firms and their market share.*

4 Transition Dynamics

We next study the dynamics that arise when large firms can expand their product lines. Time is continuous and the economy starts at time $t = 0$. The range of products of firm i at time t is $r_i(t)$ for $t \geq 0$ and $r_i(0) = r_i^0$ is given.

Similarly to Klette and Kortum (2004), at every point in time firm i can invest in innovation in order to increase the number of its product lines. An investment flow of ι_i per unit time expands r_i by $\phi(\iota_i)$ units per unit time, where $\phi(\iota_i)$ is the innovation function, assumed to be increasing and concave and $\phi(0) = 0$. Furthermore, r_i depreciates at the rate θ per unit time, which randomly hits the continuum of available brands. It follows that the product span r_i satisfies the differential equation:

$$\dot{r}_i = \phi(\iota_i) - \theta r_i, \text{ for all } t \geq 0, \quad (12)$$

where we have suppressed the time index t in $\dot{r}_i$, r_i and ι_i .

The endogenous expansion of product lines plays an important role in our theory. Hsieh and Rossi-Hansberg (2020) provide evidence that it also played an important role in the business strategies of U.S. corporations in three key sectors—services, retail and wholesale—where firm growth was dominated by expansion to new locations, i.e., new product lines. Cao *et al.* (2019) make a similar argument more broadly; firms grew predominantly on the extensive margin, through new establishments that often represented new product lines.⁶

⁵This is different, for example, from Atkeson and Burstein (2008), who have a continuum of industries, each one populated by a finite number of large single-product firms, and no small firms. Nevertheless, our and their pricing formulas have common elements.

⁶Aghion *et al.* (2019) develop a model of economic growth in which the total number (measure) of product lines is constant, but a single firm can operate multiple product lines. They focus on explaining the decline in the long-run

At every point in time the firms play a two stage game. In the first stage single-product firms enter and large firms invest in innovation. Single-product firms live only one instant of time. For this reason they make profits only in this single instant. This assumption captures in extreme form the empirical property that the turnover of plants of small firms is much larger than the turnover of plants of large firms. In the second stage all firms choose prices, in the manner described in the previous section. Under the circumstances the price index P is determined by the free entry condition (8), and it remains constant as long as the cost of entry and the cost of production of the single-product firms do no change. It follows that the profit flow of large firm i is:

$$\pi_i = r_i P^\delta p_i^{-\sigma} (p_i - a_i) - \iota_i, \text{ for all } t \geq 0,$$

where P is the same at every t while π_i , r_i , p_i and ι_i change over time, and p_i is given by (6).

In this economy the state vector is $\mathbf{r} = (r_1, r_2, \dots, r_m)$, a function of time t , and the price p_i is a function of $\mathbf{r}$. Note, however, from (9) and (10) that p_i and s_i depend only on the element r_i of $\mathbf{r}$. We therefore can express the profit function as:

$$\pi_i(\iota_i, r_i) = r_i P^\delta p_i(r_i)^{-\sigma} [p_i(r_i) - a_i] - \iota_i, \text{ for all } t \geq 0, \quad (13)$$

where $p_i(r_i)$ is the price of firm i 's brands as a function of r_i . This firm's market share is also a function of r_i , $s_i(r_i)$. From (9) and (10) we obtain the elasticities of the functions $p_i(r_i)$ and $s_i(r_i)$:

$$\frac{\partial p_i}{\partial r_i} \frac{r_i}{p_i} = \frac{\beta_i}{1 + (\sigma - 1)\beta_i}, \quad (14)$$

$$\frac{\partial s_i}{\partial r_i} \frac{r_i}{s_i} = \frac{1}{1 + (\sigma - 1)\beta_i}, \quad (15)$$

where β_i is defined in (11). Note that β_i is increasing in s_i and that, due to (10), s_i is increasing in r_i . Therefore β_i is increasing in r_i . As a result, the elasticity of the price function is larger the larger is s_i while the elasticity of the market share function is smaller the larger is s_i .

Next assume that the interest rate is constant and equal to ρ . This interest rate can be derived from the assumption that individuals discount future utility flows (1) with a constant rate ρ , so that they maximize the discounted present value of utility $\int_0^\infty e^{-\rho t} u(t) dt$. Under these circumstances firm i maximizes the discounted present value of its profits net of investment costs. It therefore solves the following optimal control problem:

growth rate. The key trigger of their dynamics is a decline in a static cost function $c(n)$ that describes a firm's overhead cost of operating n product lines. They argue that a fall in these costs was caused by the IT revolution. Their firms have constant unit costs and the quality of products can be improved by investing in innovation, as in standard models of endogenous growth with quality ladders (see Grossman and Helpman (1991) and Aghion and Howitt (1992)). A firm acquires new product lines by gaining leadership positions through quality competition. They characterize a steady state of an economy with two types of firms—high- and low-productivity (unit labor requirements)—and study the impact of a decline in $c(n)$ on concentration, labor shares, the reallocation of market shares and the long-run growth rate.

$$\max_{\{\iota_i(t), r_i(t)\}_{t \geq 0}} \int_0^\infty e^{-\rho t} \pi_i [\iota_i(t), r_i(t)] dt$$

subject to (12), (13), $r_i(0) = r_i^0$, and a transversality condition to be described below. In this problem ι_i is a control variable while r_i is a state variable. The current-value Hamiltonian of this problem is:

$$\mathcal{H}(\iota_i, r_i, \lambda_i) = \left\{ r_i P^\delta p_i(r_i)^{-\sigma} [p_i(r_i) - a_i] - \iota_i \right\} + \lambda_i [\phi(\iota_i) - \theta r_i],$$

where λ_i is the co-state variable of constraint (12). The co-state variable λ_i varies over time. The first-order conditions of this optimal control problem are:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \iota_i} &= -1 + \lambda_i \phi'(\iota_i) = 0, \\ -\frac{\partial \mathcal{H}}{\partial r_i} &= -\frac{\partial [r_i P^\delta p_i^{-\sigma}(p_i - a_i)]}{\partial r_i} + \theta \lambda_i = \dot{\lambda}_i - \rho \lambda_i, \end{aligned}$$

and the transversality condition is:

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_i(t) r_i(t) = 0.$$

In addition, the optimal path of (ι_i, r_i) has to satisfy the differential equation (12).

The above first-order conditions can be expressed as:

$$\lambda_i \phi'(\iota_i) = 1, \quad (16)$$

$$\dot{\lambda}_i = (\rho + \theta) \lambda_i - P^\delta p_i(r_i)^{-\sigma} \left\{ p_i(r_i) - a_i - r_i \left(\sigma p_i(r_i)^{-1} [p_i(r_i) - a_i] - 1 \right) p'(r_i) \right\}. \quad (17)$$

From (16) we obtain the investment level ι_i as an increasing function of λ_i , which we represent as $\iota_i(\lambda_i)$. Substituting this function into (12) yields the autonomous differential equation:

$$\dot{r}_i = \phi[\iota_i(\lambda_i)] - \theta r_i. \quad (18)$$

Next we substitute (6), (11) and (14) into (17) to obtain a second autonomous differential equation:

$$\dot{\lambda}_i = (\rho + \theta) \lambda_i - \Gamma_i(r_i), \quad (19)$$

where:

$$\Gamma_i(r_i) \equiv a_i^{1-\sigma} P^\delta \sigma \left[\frac{\sigma - \delta s_i(r_i)}{\sigma - \delta s_i(r_i) - 1} \right]^{-\sigma} \frac{1}{[\sigma - \delta s_i(r_i) - 1] \sigma + s_i(r_i)^2 \delta^2}. \quad (20)$$

The function $\Gamma_i(r_i)$ represents the profitability of a new product line, given the firm's product span r_i ; that is, it represents the marginal profitability of r_i . We show in the Appendix that this marginal profitability declines in r_i , i.e., $\Gamma'_i(r_i) < 0$.

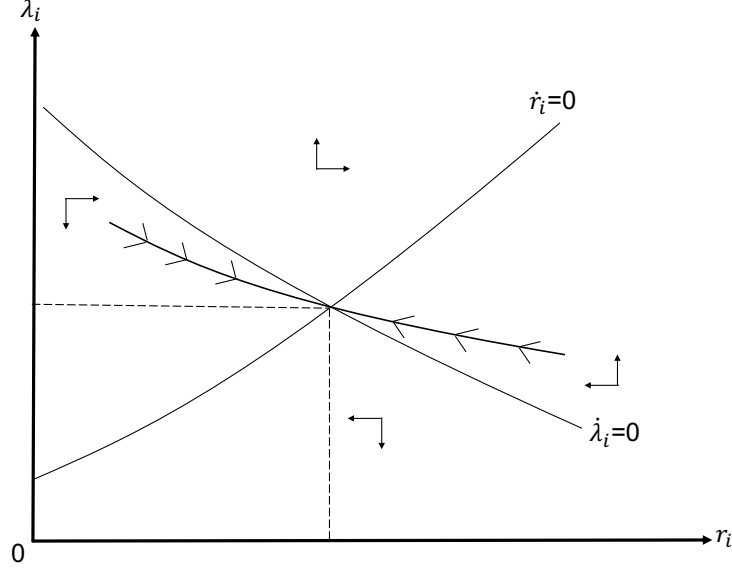


Figure 1: Transition Dynamics

A solution to the autonomous system of differential equations (18) and (19) that satisfies the transversality condition is also a solution to the firm's optimal control problem, because $\mathcal{H}(\iota_i, r_i, \lambda_i)$ is concave in the first two arguments. This can be seen by observing that the Hamiltonian is additively separable in ι_i and r_i , and it is strictly concave in ι_i and in r_i . The steady state of these differential equations is characterized by:

$$\phi[\iota_i(\lambda_i)] = \theta r_i, \quad (21)$$

$$(\rho + \theta) \lambda_i = \Gamma_i(r_i). \quad (22)$$

The left-hand side of (21) is an increasing function of λ_i . Therefore the curve in (r_i, λ_i) space along which r_i is constant is upward sloping. The right-hand side of (22) is declining in r_i , because $\Gamma'_i(r_i) < 0$. Therefore the curve in (r_i, λ_i) space along which λ_i is constant is downward sloping. These curves are depicted in Figure 1. Based on the differential equations (18)-(19), the figure also depicts the resulting dynamics. There is a single stable saddle-path along which (r_i, λ_i) converge to the steady state and the transversality condition is satisfied in this steady state. On this saddle path either r_i rises and λ_i declines or r_i declines and λ_i rises, depending on whether r_i^0 is below or above its steady-state value.

Now suppose that all the r_i^0 s are below their steady state values (this case arises, for example, when the economy is in steady state and innovation costs decline; see below). Then every large firm expands its range of products over time. As a result, the number of single-product firms shrinks. This process continues until the economy reaches a steady state.

If at some point in time the number of single-product firms drops to zero, the dynamics change.⁷

⁷From that point on the optimal strategy of large firm i depends on the entire state vector $\mathbf{r}$. As a result, the firms

We focus, however, on the case in which $\bar{r} > 0$ for all $t \geq 0$. In this case the price index P remains constant as long as f and $\bar{a}$ do not change.

What can be said about the dynamics of profits of a large firm i ? Changes of profits over time can be expressed as:

$$\frac{\partial \pi_i(\iota_i, r_i)}{\partial t} = -\frac{\partial \iota_i}{\partial t} + \frac{\partial [r_i P^\delta p_i^{-\sigma} (p_i - a_i)]}{\partial r_i} \frac{\partial r_i}{\partial t}.$$

From (16) we see that ι_i is an increasing function of λ_i , and since for a firm that expands its product range λ_i declines, it follows that the investment level ι_i also declines over time, raising profits net of investment costs. From (17) and (19) we see that:

$$\frac{\partial [r_i P^\delta p_i^{-\sigma} (p_i - a_i)]}{\partial r_i} = \Gamma_i(r_i) > 0.$$

Therefore, profits grow in a firm that adds new product lines. We conclude that profits net of investment costs rise in growing firms. Since wages are constant, this implies that in an economy in which all large firms grow, the share of labor in national income declines.

While the share of labor declines over time, average markups are rising. To see the sources of variation in average markups, note that the average markup μ_{av} can be expressed as a weighted average of the markups of all single- and multi-product firms:

$$\mu_{av} = \left(1 - \sum_{i=1}^m s_i\right) \bar{\mu} + \sum_{i=1}^m s_i \mu_i,$$

where the markup of a single product firm is $\bar{\mu} = \sigma / (\sigma - 1)$ and the markup of large firm i is $\mu_i = (\sigma - \delta s_i) / (\sigma - \delta s_i - 1)$. Since the markup of every large firm is higher than the markup of every single-product firm and the market share of every large firm rises over time, the average markup also increases. The increase in the average markup is driven by two forces: rising markups of the large firms and market share reallocation from low-markup (single-product) to high-markup (multi-product) firms.⁸ We summarize these findings in

engage in a differential game. Since no firm can commit to the entire path of its investments ι_i , one needs to adopt the closed loop solution to this game, in which the investment level ι_i is a function of the state vector $\mathbf{r}$. There do not exist user-friendly characterizations of solutions to such games. Instead, we provide in the Appendix an analysis of the impact of changes in the state variables r_i on prices, markups and market shares of the large firms.

⁸Using a dynamic model of monopolistic competition with a Kimball aggregator, Edmond *et al.* (2019) decompose the welfare cost of markups into three sources of influence: (i) aggregate markup; (ii) misallocation of inputs; and (iii) inefficiently low entry of firms. Their quantitative model implies that (i) accounts for $3/4$ of the welfare cost while (ii) accounts for $1/4$. The impact of entry is negligible. They show that in the Compustat data the sales-weighted aggregate markup is higher than the cost-weighted aggregate markup and that the gap between them has widened over time (see their Figure 8). In their model the cost-weighted aggregate markup turns out to be the relevant measure for (i). Our aggregate markup μ_{av} has been constructed using sales shares, which seems to be suited for our purpose. Nevertheless, we discuss in the Appendix a cost-weighted measure of the aggregate markup:

$$\mu_{av}^c = \left(1 - \sum_{i=1}^m \varrho_i\right) \bar{\mu} + \sum_{i=1}^m \varrho_i \mu_i,$$

where ϱ_i is the variable cost share of firm i . This measure does not necessarily rise on the transition path, but neither does it play a distinct role in our analysis. We show in the Appendix that the cost weight ϱ_i is rising in r_i if and

Proposition 3. *Consider an economy in which the initial range of products r_i^0 is smaller than its steady state value for every i , and in which $\bar{r} > 0$ at all times. Then over time: (i) every large firm i widens its product span, raises its markup, and experiences rising profits net of investment costs; (ii) the average markup rises and the labor share declines.*

Since wages are constant, so is wage income. Nonetheless, in view of Proposition 3(i), aggregate income—which consists of labor income plus aggregate profits net of investment costs—is rising during the transition to a steady state. In view of the indirect utility function (3), this implies that aggregate utility rises over time (recall that P remains constant). Moreover, if this economy is populated by some individuals who own shares in large firms and other individuals who do not, the growth of large multi-product firms widens the disparity of well-being between these two groups. We summarize this finding in

Corollary 1. *Consider an economy in which the initial range of products r_i^0 is smaller than its steady state value for every i , and in which $\bar{r} > 0$ at all times. Then over time the well-being of individuals who derive income from labor only is constant while the well-being of individuals who own shares in large firms rises.*

5 Comparative Dynamics

For an economy that is initially in steady state, we study in this section the dynamics that arise in response to changes in the cost of inventing new product lines, the marginal costs of production and the cost of entry of single-product firms.

First, consider a change in the cost of innovation, as reflected in a shift of the function $\phi(\iota_i)$. We take $\kappa > 0$ to be a productivity measure of innovation and express the modified innovation function as $\kappa\phi(\iota_i)$. Initially $\kappa = 1$. An upward shift in κ represents a rise in the productivity of investment in innovation or a decline in innovation costs, while a decline in κ represents a decline in the productivity of investment in innovation or a rise in innovation costs. The latter may arise when it becomes harder to invent new product lines. With the new innovation function the dynamics of product span, (12), become:

$$\dot{r}_i = \kappa\phi(\iota_i) - \theta r_i, \text{ for all } t \geq 0. \quad (23)$$

In this case the first-order condition of the optimal control problem (16) becomes:

$$\lambda_i \kappa \phi'(\iota_i) = 1, \quad (24)$$

while the differential equation (19) does not change. From (24) we obtain the investment level ι_i as an increasing function of $\kappa\lambda_i$, which we express as $\iota_i(\kappa\lambda_i)$. This is the same $\iota_i(\cdot)$ function that we

only if $\beta_i < 1$ and it is declining in r_j , $j \neq i$, if and only if $\beta_j < 1$. As a result, μ_{av}^c is rising due to increases in the markups of the large firms, μ_i , $i = 1, 2, \dots, m$, but the shifts in the cost shares may bring it down.

had before. Substituting this function into (23) yields the autonomous differential equation:

$$\dot{r}_i = \kappa \phi [\iota_i (\kappa \lambda_i)] - \theta r_i.$$

The steady state of this differential equations is characterized by:

$$\kappa \phi [\iota_i (\kappa \lambda_i)] = \theta r_i,$$

while the second steady state equation, (22), does not change, because the differential equation (19) remains the same. For $\kappa = 1$, the steady state and the dynamics depicted in Figure 1 remain the same.

Now consider an increase in κ , which represents a decline in the costs of inventing new product lines. Since $\kappa \phi [\iota_i (\kappa \lambda_i)]$ is increasing in κ , this leads to a downward shift of the $\dot{r}_i = 0$ curve without changing the $\dot{\lambda}_i = 0$ curve. As a result, λ_i declines on impact to a new saddle path, starting transition dynamics with declining values of λ_i and rising values of r_i . This process takes place in every large firm, leading to a new steady state in which every large firm has a larger product span, a larger market share and a higher markup. The average markup rises during the transition and it is higher in the new steady state. The flow of aggregate utility also rises during this transition and is higher in the new steady state. The flow utility rises because profits net of investment costs rise while the price index P remains the same. We therefore have

Proposition 4. *Suppose that every large firm i is in steady state and $\bar{r} > 0$ at all times. Then a decline in the cost of innovation, i.e., an increase in κ , leads all large firms to expand their product range, raise their market shares and raise their markups. Contemporaneously, average markups increase and so does the aggregate flow of utility.*

We next turn to changes in the marginal costs of production and the cost of entry of single-product firms. As is evident from (21) and (22), such changes impact the new steady state through the function $\Gamma_i(r_i)$ only. A change that raises $\Gamma_i(r_i)$ shifts upward the $\dot{\lambda}_i = 0$ curve in Figure 1. After the impact effect, which results from the upward jump in λ_i , the dynamic process leads to a gradual widening of the span of products and increases in the markup and profits net of investment costs. In contrast, a change that reduces $\Gamma_i(r_i)$ shifts downward the $\dot{\lambda}_i = 0$ curve. After the downward jump of λ_i on impact, the dynamic process then leads to a gradual narrowing of the span of products and declines in markups and profits net of investment costs.

First, consider a decline in a_i , resulting from a technical improvement in the firm's technology. We show in the Appendix that the impact of a_i on Γ_i can be expressed as:

$$\begin{aligned} \hat{\Gamma}_i &= -(\sigma - 1) \hat{a}_i + \left(\frac{\partial \Gamma_i}{\partial s_i} \frac{s_i}{\Gamma_i} \right) \left(\frac{\partial s_i}{\partial a_i} \frac{a_i}{s_i} \right) \hat{a}_i \\ &= \frac{(\sigma - 1) s_i^2 \delta^2 - (\sigma - \delta s_i - 1)^2 (\sigma^2 - \delta^2 s_i^2)}{[(\sigma - \delta s_i - 1) \sigma + s_i^2 \delta^2]^2} (\sigma - 1) \hat{a}_i. \end{aligned} \tag{25}$$

The relationship between a_i and Γ_i portrayed by this equation does not depend on the cost structure of other firms. Moreover, it implies that a decline in a_i shifts upward the $\dot{\lambda}_i = 0$ curve if and only if:

$$(\sigma - \delta s_i - 1)^2 (\sigma^2 - \delta^2 s_i^2) > (\sigma - 1) s_i^2 \delta^2. \quad (26)$$

The potential ambiguity of the response of Γ_i to changes in a_i results from the existence of two channels through which the marginal cost impacts the profitability of a new variety (the marginal profitability of r_i), as can be seen from (20). A decline in a_i raises Γ_i for a given market share s_i , due to cost savings in production. But, as shown in (10), a decline in a_i raises the market share of firm i and a rise in the firm's market share reduces the profitability of a new variety. It follows that the shift of the $\dot{\lambda}_i = 0$ curve depends on the strength of these two effects: if the response of the market share dominates, the curve shifts down; and if the response of the market share does not dominate, the curve shifts up. The strength of the market share effect depends in turn on the firm's initial size. For low values of s_i the impact through the market share channel is small, and (26) is satisfied. But (26) is less likely to be satisfied the larger s_i is, because the left-hand side of this inequality is declining in s_i while the right-hand side is increasing. This leads to the following

Lemma 1. *If $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) > (\sigma - 1) \delta^2$, then (26) is satisfied for all market shares $s_i \in [0, 1]$. And if $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) < (\sigma - 1) \delta^2$, then there exists a market share $s^o \in (0, 1)$, defined by:*

$$(\sigma - \delta s^o - 1)^2 [\sigma^2 - \delta^2 (s^o)^2] = (\sigma - 1) (s^o)^2 \delta^2,$$

such that (26) is satisfied for $s_i < s^o$ and violated for $s_i > s^o$.

Given the assumption $\sigma > \varepsilon > 1$, the inequality $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) > (\sigma - 1) \delta^2$ is satisfied when ε is close to σ and violated when ε is close to one (recall that $\delta = \sigma - \varepsilon$). We therefore have

Proposition 5. *Suppose that firm i is in steady state and $\bar{r} > 0$ at all times. Then a decline in a_i triggers an adjustment process that gradually raises r_i as well as i 's markup and profits net of investment costs if either $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) > (\sigma - 1) \delta^2$ or $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) < (\sigma - 1) \delta^2$ and $s_i < s^o$, where s^o is defined in Lemma 1. Otherwise, this technical improvement triggers an adjustment process that gradually reduces r_i while i 's markup and profits net of investment costs decline gradually after increasing on impact.*

Using these results, we can examine the dynamics of firm i 's market share. Since on impact the span of products does not change (r_i is a state variable), (10) implies that the decline in the marginal cost raises on impact firm i 's market share. Moreover, if the adjustment process leads to a gradual expansion of its product span, i 's market share rises over time until it reaches a new steady state. In this case the firm has a larger market share in the new steady state. If, however, the adjustment process leads to a narrowing of the firm's product span, then (10) implies that the initial upward jump in firm i 's market share is followed by a gradual decline in its market share. A

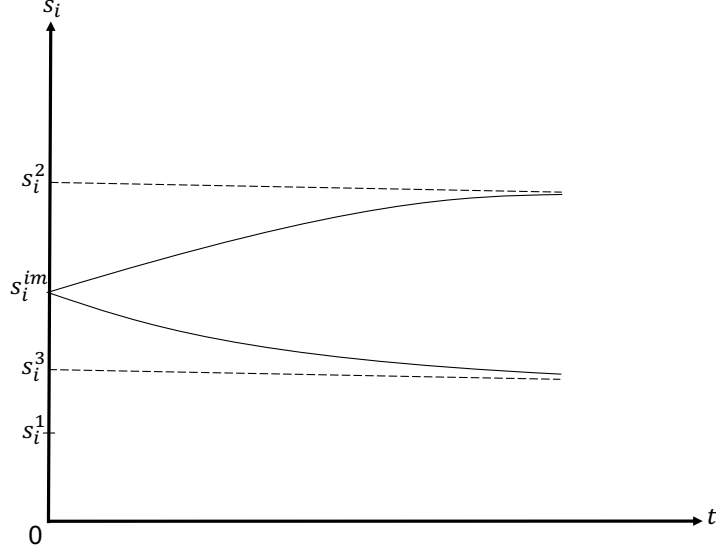


Figure 2: Dynamics of the market share in response to a decline in the marginal cost a_i

question then arises whether this firm's market share is larger or smaller in the new steady state. We prove the following

Proposition 6. *Suppose that firm i is in steady state and $\bar{r} > 0$ at all times. Then a decline in a_i triggers an adjustment process that raises s_i in the new steady state.*

Proof. We have shown that the market share is larger in the new steady state when the adjustment process involves expansion of the firm's product span. It therefore remains to show that this is also true when the adjustment process involves contraction of the product span. To this end note that a decline in r_i on the transition path is triggered by a decline in the marginal profitability of r_i in response to a decline in a_i , which leads in turn to a downward shift in the $\dot{\lambda}_i = 0$ curve in Figure 1. In this case the new steady state has a lower r_i as well as a lower λ_i . Next note from the steady state condition (22) that a lower λ_i implies a lower Γ_i . Recall, however, that for a constant s_i a fall in a_i raises Γ_i , and therefore Γ_i can be lower in the new steady state only if s_i is higher. In sum, independently of whether a decline a_i shifts upward or downward the $\dot{\lambda}_i = 0$ curve, the market share s_i is larger in the new steady state. $\square$

This result yields the following

Corollary 2. *Consider an economy in steady state with active single-product firms. Then large firms with lower marginal costs have larger market shares.*

The dynamic patterns of the market share that have been uncovered by this analysis are depicted in Figure 2, where s_i^1 is the market share in the initial steady state. First, the market share jumps up to s_i^{im} on impact when a_i declines. Afterward, the market share rises continuously until it reaches s_i^2 , as portrayed by the upper curve, or it declines continuously until it reaches s_i^3 , as portrayed

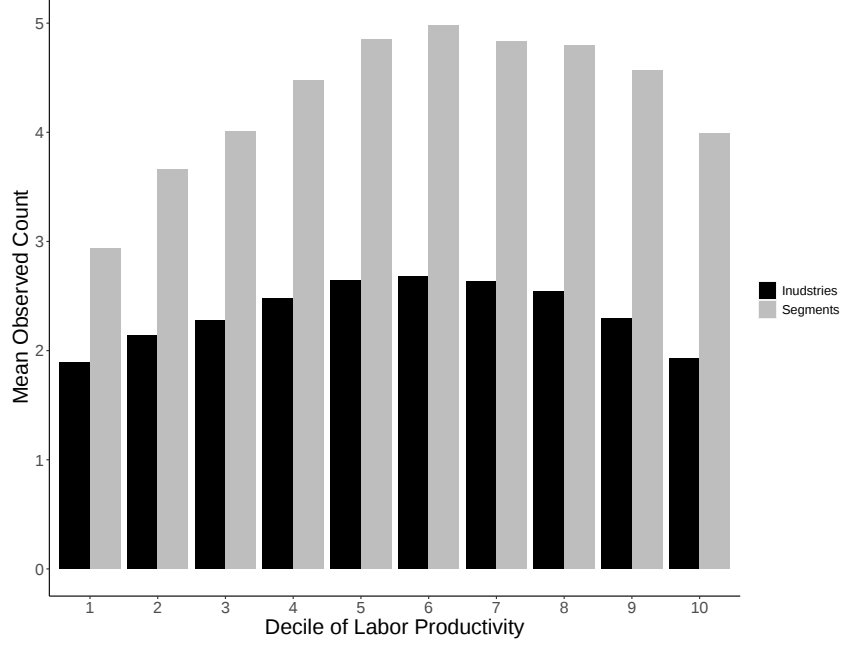


Figure 3: Average Number of Product Lines vs. Labor Productivity Deciles

by the lower curve. In both cases the new steady state market share exceeds s_i^1 . The former case applies when $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) > (\sigma - 1) \delta^2$ or $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) < (\sigma - 1) \delta^2$ and $s_i^1 < s^o$, and the latter case applies otherwise.

These results suggest three possible steady state patterns for the relationship between a_i and r_i in the cross section of multi-product firms: lower-cost firms have larger product spans, lower-cost firms have smaller product spans, or the relationship between marginal costs and product spans has an inverted U shape. The first pattern holds for all marginal cost structures when $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) > (\sigma - 1) \delta^2$. In the opposite case, when $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) < (\sigma - 1) \delta^2$, there exist high values of a_i at which $s_i < s^o$, and among firms with such high marginal costs firms with lower marginal costs have larger product spans. Moreover, there exist low values of a_i at which $s_i > s^o$, and among firms with such low marginal costs lower-cost firms have smaller product spans. Combining these results we have

Proposition 7. *Consider an economy in steady state with active single-product firms. Then, in the cross section of multi-product firms r_i is declining in s_i , rising in s_i , or rising in s_i among firms with low market shares and declining in s_i among firms with high market shares.*

Combining this Proposition with Corollary 2, we note that our model raises the possibility of an inverted-U relationship between labor productivity, as measured by $1/a_i$, and the number of product lines, r_i . We now show that this prediction is not only a theoretical possibility, but that there is suggestive evidence for such a relationship in the Compustat data set. To this end we collected data on revenue, employment, the number of sectors in which a firm operated and the number of segments in which a firm operated, all for 2018. We computed labor productivity as revenue per

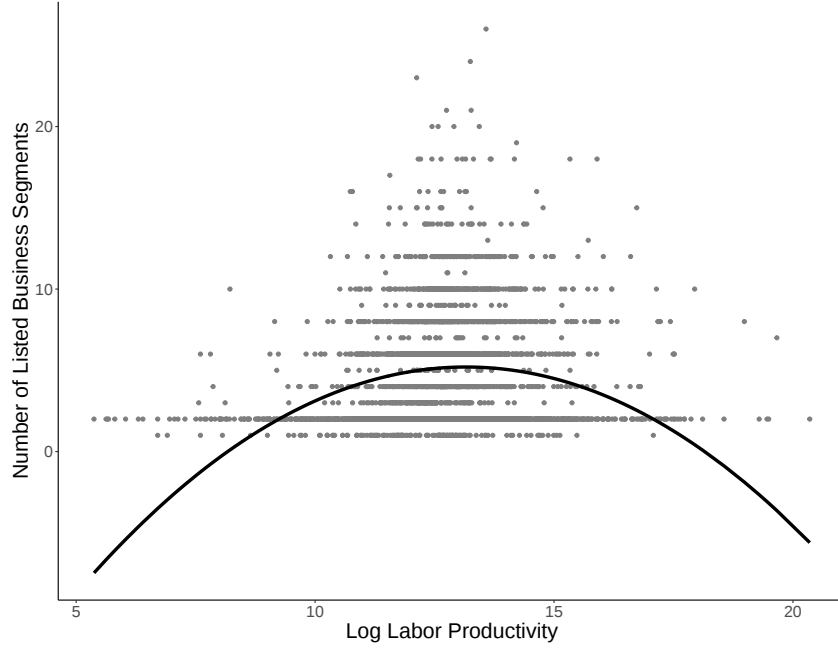


Figure 4: Number of Segments vs. Labor Productivity

worker and we treat the number of segments as a proxy for the number of product lines. As a robustness check, we also consider the number of industries in which a firm operated as a proxy for the number of its product lines.⁹ Figure 3 depicts the relationships between our two proxies for r_i and our proxy for $1/a_i$. On the horizontal axis the firms are divided into deciles, based on their labor productivity. On the vertical axis we report the mean number of segments and the mean number of industries in each decile. As is evident, these relationships exhibit an inverted-U.

To further examine these relationships, we regressed the number of segments or the number of sectors in which a firm operates on a second-order polynomial of the log of labor productivity. We report in the Appendix the resulting OLS estimates. The coefficient on log labor productivity is positive and the coefficient on the log of labor productivity squared is negative in both case. Moreover, all four coefficients are significantly different from zero. Figure 4 plots the data points that we have used (more than 4,000 observations) as well as the fitted quadratic curve. The first thing to note is that there are many firms with similar numbers of segments and different labor productivity levels, especially when the number of segments is low. Nevertheless, the estimated curve has the shape of an inverted-U. We report in the Appendix a similar graph for the number of industries in which a company operates. In conclusion, while we view this paper as a theoretical contribution, we have also provided suggestive evidence for the inverted-U curve predicted by our model.

⁹About 70% of the firms in the Compustat database breakdown the company into segments. Segments include different business lines or geographic locations. We use a company's number of business segments as a proxy for the number of product lines. Within each segment, the firm can list up to two SIC codes in which the business segment operates. The total number of unique SIC codes listed across business segments is what we define as the number of industries in which a firm operates. This is our second proxy for the number of product lines.

5.1 Costs of Single-Product Firms

We next examine the impact of the cost structure of single-product firms. As is evident from (8), a decline in either the marginal cost or the entry cost of single-product firms reduces the price index P , thereby raising the competitive pressure in the economy. How do the large firms respond to this rise in competition? To answer the question, suppose that all firms are in steady state. Equation (20) implies:

$$\hat{\Gamma}_i = \delta \hat{P}_i + \left(\frac{\partial \Gamma_i}{\partial s_i} \frac{s_i}{\Gamma_i} \right) \left(\frac{\partial s_i}{\partial P} \frac{P}{s_i} \right) \hat{P}. \quad (27)$$

A decline in the price index P , which elevates the competitive pressure on every large firm, reduces the marginal value of r_i . For this reason the first term on the right-hand side of this equation is negative when $\hat{P}_i < 0$. In response, firm i reduces its price and market share (see (9) and (10)) and the fall in market share raises the marginal value of r_i . For this reason the second term on the right-hand side is positive when $\hat{P}_i < 0$. It follows that a decline in P shifts the $\dot{\lambda}_i = 0$ curve downward in Figure 1 if the competition effect dominates and upward if the market share effect dominates. Using (10), it is evident that for $\varepsilon \rightarrow 1$ (27) is similar to (25), except for the opposite sign on their right-hand sides. Therefore, in this case a decline in P shifts down the $\dot{\lambda}_i = 0$ curve if and only if a decline in a_i shifts it up. Under these conditions a lower P may lead to a lower or higher value of r_i in steady state, and moreover, its impact may vary across firms with different marginal costs and therefore different market shares s_i . For $\varepsilon \rightarrow 1$ the inequality $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) > (\sigma - 1) \delta^2$ is violated, implying that there exists an s_P^o such that the decline in P shifts the $\dot{\lambda}_i = 0$ curve down for $s_i < s_P^o$ and up for $s_i > s_P^o$. Therefore, in this case a rise in the competitive pressure shrinks the product span of multi-product firms with $s_i < s_P^o$ and expands the product span of multi-product firms with $s_i > s_P^o$. As a result, the gaps in market shares between large and small multi-product firms widens, thereby increasing the inequality in the size distribution of firms.¹⁰ Alternatively, for $\varepsilon \rightarrow \sigma$, the competition effect is negligible and the shift in the market share dominates the impact on Γ_i . As a result, the $\dot{\lambda}_i = 0$ curve shifts up for all multi-product firms, raising their product spans.

Finally, note that a decline in P reduces the steady state market share of every large firm. This is clearly the case when every firm's product span declines because in this case both P and r_i diminish the market share (see (10)). Alternatively, for a firm that expands its steady state r_i , the value of λ_i is higher in the new steady state (see (21)). Therefore this firm's Γ_i is also larger in the new steady state (see (22)). But the direct impact of the decline in P on Γ_i is negative, and therefore s_i has to be smaller for Γ_i to be larger. We therefore have

Proposition 8. *Consider an economy in steady state with $\bar{r} > 0$ at all times. Then, a technical improvement that reduces either f or $\bar{a}$ may raise r_i in the new steady state for all i , reduce r_i for all i , or reduce r_i of the small multi-product firms and raise r_i of the large multi-product firms. Nevertheless, s_i is smaller in the new steady state for all large firm i .*

Figure 5 depicts the dynamics of two firms, i and j , for the case in which $s_i < s_P^o$ and $s_j > s_P^o$,

¹⁰From (10), $\hat{s}_i - \hat{s}_j = (\hat{r}_i - \hat{r}_j) / [1 + (\sigma - 1) \beta_i]$. Therefore $\hat{s}_i > \hat{s}_j$ if and only if $\hat{r}_i > \hat{r}_j$.

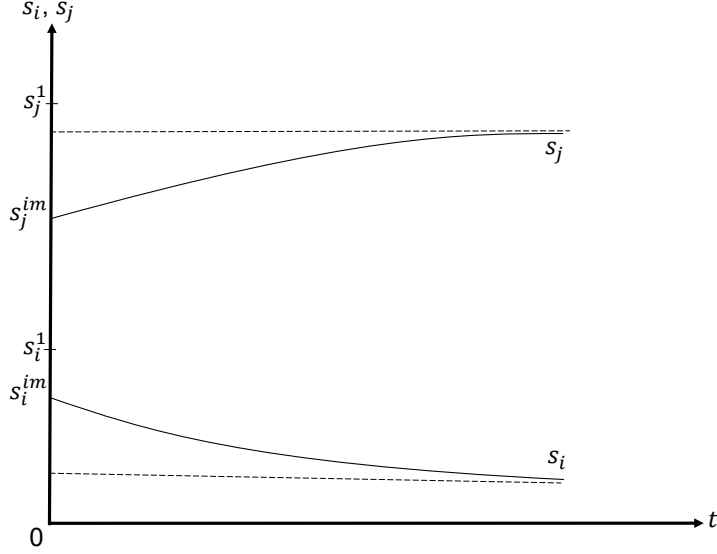


Figure 5: Dynamics of market shares in response to a decline in P

where s_P^o is the cutoff market share for the opposite firm dynamics. Firm i starts with $s_i = s_i^1$ while firm j starts with $s_j = s_j^1$. In both firms the market share jumps down on impact as a result of the decline in P , to s_i^{im} and s_j^{im} , respectively. After that, the market share of the smaller firm declines while the market share of the larger firm rises. Yet in both cases, the market share is lower in the new steady state.

Gutierrez and Philippon (2019) find that the elasticity of the number of firms with respect to Tobin's Q declined during 1995-2010. They argue that this resulted from increased entry costs due to regulation rather than due to technological developments or financial frictions. In our model an increase in f generates the above described dynamics independently of the source of variation in the fixed cost of entry. According to Proposition 8, an increase in f raises the long-run market share of all large multi-product firms and reduces the joint market share of the small single product firms. Yet, it may have an uneven impact on the span of products of the large firms. That is, it may increase the number of product lines of the smaller multi-product firms and reduce the number of product lines of the large ones, thereby flattening the relationship between labor productivity (i.e., $1/a_i$) and product span.

6 Conclusion

We have developed a parsimonious model of industry evolution, in which large multi-product firms grow via investment in new product lines. While these firms are oligopolies, they face competitive pressure from small single-product firms that engage in monopolistic competition. These features accord with the evidence discussed in the introduction. Our model generates time patterns of markups, concentration, and labor shares that are consistent with the data. Moreover, it predicts

rich patterns for the cross-section of firms. In particular, it predicts an inverted-U relation between labor productivity and product span, for which we provide supportive evidence. It also predicts that rising competitive pressure from small single-product firms flattens the cross-sectional relationship between labor productivity and product span among the large multi-product firms. Although this study consists of a theoretical contribution, we believe that our model delivers valuable insights into industry dynamics that can be empirically studied. There are few data sets containing information on product span of individual firms, and these data are mostly confidential. Nevertheless, we hope that the predictions of our model will eventually be examined with some of the existing rich data sets. Finally, we show in the appendix how to construct an aggregate economy with a continuum of industries of the type studied in this paper. This type of economy can be used to study various macroeconomic issues, including economic growth.

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Appendix

Comparative Dynamics

We first derive the slope of the $\dot{\lambda}_i=0$ curve. Differentiation of the right-hand side of (22) yields:

$$\hat{\Gamma}_i = -(\sigma - 1)\hat{a}_i + \delta\hat{P} - \frac{\sigma\delta s_i}{(\sigma - \delta s_i - 1)(\sigma - \delta s_i)}\hat{s}_i + \frac{\delta s_i(\sigma - 2\delta s_i)}{(\sigma - \delta s_i - 1)\sigma + \delta^2 s_i^2}\hat{s}_i.$$

This equation implies that the right-hand side of (22) is declining in r_i because Γ_i is declining in s_i and s_i is rising in r_i (see (10)). The former is seen from this equation by observing that $\sigma\delta s_i > \delta s_i(\sigma - 2\delta s_i)$ and $(\sigma - \delta s_i - 1)(\sigma - \delta s_i) < (\sigma - \delta s_i - 1)\sigma + \delta^2 s_i^2$. Collecting terms we can rewrite this equation as:

$$\hat{\Gamma}_i = -(\sigma - 1)\hat{a}_i + \delta\hat{P} - \delta^2 s_i^2 \frac{2(\sigma - \delta s_i - 1)(\sigma - \delta s_i) + \sigma(\sigma - 1)}{(\sigma - \delta s_i - 1)(\sigma - \delta s_i)[(\sigma - \delta s_i - 1)\sigma + \delta^2 s_i^2]}\hat{s}_i. \quad (28)$$

Next consider the total effect of a shift in the marginal cost a_i on Γ_i . From (10) we have:

$$\hat{s}_i = -\frac{\sigma - 1}{1 + (\sigma - 1)\beta_i}\hat{a}_i = -\frac{(\sigma - 1)(\sigma - \delta s_i - 1)(\sigma - \delta s_i)}{(\sigma - \delta s_i - 1)(\sigma - \delta s_i) + (\sigma - 1)\delta s_i}\hat{a}_i.$$

Substituting this expression into (28) we obtain the total impact of a_i on Γ_i :

$$\begin{aligned} \frac{\hat{\Gamma}_i}{(\sigma - 1)\hat{a}_i} &= -1 + \delta^2 s_i^2 \frac{2(\sigma - \delta s_i - 1)(\sigma - \delta s_i) + \sigma(\sigma - 1)}{[(\sigma - \delta s_i - 1)\sigma + s_i^2\delta^2]^2} \\ &= \frac{(\sigma - 1)s_i^2\delta^2 - (\sigma - \delta s_i - 1)^2(\sigma^2 - \delta^2 s_i^2)}{[(\sigma - \delta s_i - 1)\sigma + s_i^2\delta^2]^2}. \end{aligned}$$

It follows that a decline in the marginal cost a_i shifts upward the $\dot{\lambda}_i=0$ curve if and only if $(\sigma - 1)s_i^2\delta^2 < (\sigma - \delta s_i - 1)^2(\sigma^2 - \delta^2 s_i^2)$.

Now consider the cost-weighted average markup:

$$\mu_{av}^c = \left(1 - \sum_{i=1}^m \varrho_i\right) \bar{\mu} + \sum_{i=1}^m \varrho_i \mu_i,$$

where ϱ_i is the variable cost share of firm i . The variable cost of firm i is:

$$\begin{aligned} a_i r_i x_i &= a_i r_i P^\delta p_i^{-\sigma} \\ &= a_i^{1-\sigma} r_i P^\delta \left(\frac{\sigma - \delta s_i}{\sigma - \delta s_i - 1}\right)^{-\sigma} \end{aligned}$$

and the variable costs of the small firms are:

$$\begin{aligned}\bar{a}\bar{r}\bar{x} &= \bar{a}\bar{r}P^\delta\bar{p}^{-\sigma} \\ &= \bar{a}^{1-\sigma}\bar{r}P^\delta\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma}.\end{aligned}$$

Therefore:

$$\begin{aligned}\varrho_i &= \frac{a_i^{1-\sigma}r_i\mu_i^{-\sigma}}{\bar{a}^{1-\sigma}\bar{r}\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} + \sum_{j=1}^m a_j^{1-\sigma}r_j\mu_j^{-\sigma}} \\ &= \frac{a_i^{1-\sigma}r_i\left(\frac{\sigma-\delta s_i}{\sigma-\delta s_i-1}\right)^{-\sigma}}{\bar{a}^{1-\sigma}\bar{r}\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} + \sum_{j=1}^m a_j^{1-\sigma}r_j\left(\frac{\sigma-\delta s_j}{\sigma-\delta s_j-1}\right)^{-\sigma}}.\end{aligned}$$

On the dynamic path with a constant P , the market share of firm i rises with r_i according to (see (10) and (11)):

$$\hat{s}_i = \frac{1}{1 + (\sigma - 1)\beta_i}\hat{r}_i,$$

where:

$$\beta_i = \frac{\delta s_i}{(\sigma - \delta s_i - 1)(\sigma - \delta s_i)} > 0.$$

It follows that:

$$\begin{aligned}\frac{d(r_i\mu_i^{-\sigma})}{r_i\mu_i^{-\sigma}} &= \frac{d\left[r_i\left(\frac{\sigma-\delta s_i}{\sigma-\delta s_i-1}\right)^{-\sigma}\right]}{r_i\left(\frac{\sigma-\delta s_i}{\sigma-\delta s_i-1}\right)^{-\sigma}} \\ &= \hat{r}_i - \sigma\beta_i\hat{s}_i \\ &= \hat{r}_i \left[1 - \frac{\sigma\beta_i}{1 + (\sigma - 1)\beta_i}\right].\end{aligned}$$

The expression in the square bracket in the last line of this equation is positive if and only if $\beta_i < 1$. Therefore, during the transition with rising product spans of all large firms ϱ_i is rising in response to the increase in r_i if and only if $\beta_i < 1$ and declining in response to the increase in r_j if and only if $\beta_j < 1$. Note, however, that $\beta_i < 1$ if and only if:

$$\delta s_i < (\sigma - \delta s_i - 1)(\sigma - \delta s_i),$$

or:

$$(\sigma - \delta s_i)^2 > \sigma.$$

Using $\delta = \sigma - \varepsilon$, this implies that $\beta_j < 1$ if and only if:

$$s_i \varepsilon + (1 - s_i) \sigma > \sqrt{\sigma}.$$

This inequality is always satisfied for $\varepsilon > \sqrt{\sigma}$.

Table 1: Average Number of Product Lines vs. Productivity Deciles

<i>Decile</i>	<i>Log(Prod)</i>	<i>MeanInd</i>	<i>MeanSegs</i>
1	10.05	1.89	2.93
2	11.54	2.14	3.65
3	12.04	2.27	4.00
4	12.31	2.48	4.47
5	12.54	2.64	4.84
6	12.77	2.67	4.98
7	13.06	2.63	4.83
8	13.42	2.53	4.79
9	13.91	2.29	4.57
10	15.31	1.92	3.99

Note: This table shows the deciles of average log labor productivity for firms in the Compustat database for the year 2018, available through WRDS. Labor productivity is defined as the ratio of total sales to employment. It also shows the mean number of industries and business segments that are reported in the Compustat Segments Data. The data was accessed on June 2, 2020.

Table 2: Quadratic Relationship of Productivity on Product Span

	Industries	Segments
log(Prod)	2.85*** (0.45)	5.50*** (1.02)
log(Prod) ²	-0.11*** (0.02)	-0.21*** (0.04)
Primary Ind. FE	YES	YES
Obs	4126	4126
R ²	0.7334	0.4603

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Note: This table shows the results of an OLS quadratic regression of the number of industries or segments on the log of labor productivity. The data includes all firms with positive sales and employment in the Compustat database for the year 2018. Labor productivity is defined as the ratio of total sales to employment. Segments here refers to the total number of business segments listed in the Compustat Segments Data by firm. The number of industries is the number of primary and secondary SIC codes listed across all business segments. We also include fixed effects for 4 digit primary SIC code listed on Compustat. Data was accessed on June 2, 2020.

We now provide addition information on the empirical analysis in this section. Table 1 presents the data that has been used to construct Figure 4 while Table 2 presents the regression results. As pointed out in the main text, the coefficient for log productivity is positive and significantly different from zero and the coefficient for the square of log productivity is negative and significantly different from zero in both specifications; i.e., when we use the number of industries or the number of segments to measure a firm's product span. While in the main text we reported in Figure 3 the curvature of this quadratic form for the number of segments as a proxy for the number of product lines, we now report a similar figure when the number of industries is used as a proxy for the number

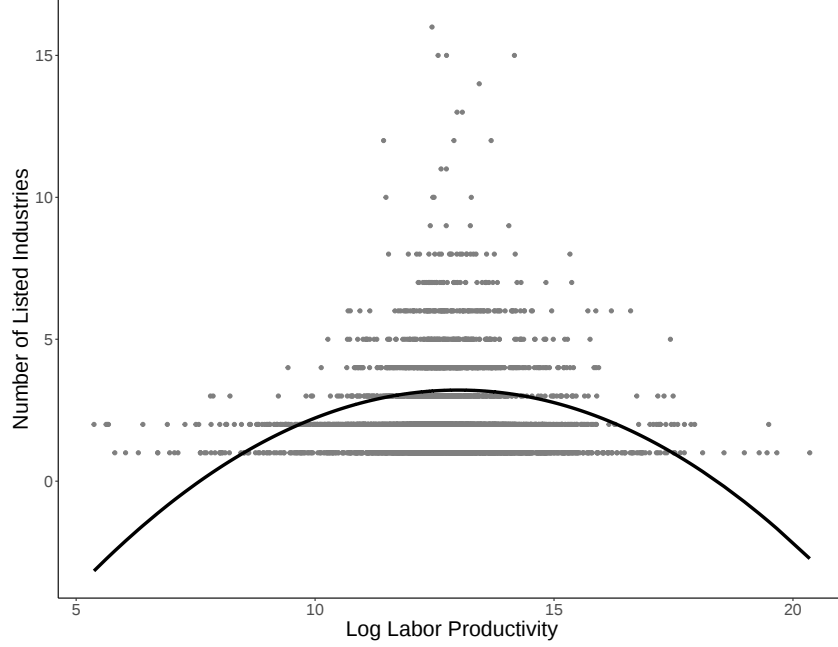


Figure 6: Number of Industries vs. Labor Productivity

of product lines. As is evident, the two figures are quite similar.

Comparative Statics: Given Number of Brands

In this section we examine the case in which the number of single-product firms, $\bar{r}$, as well the number of products available to each one of the large firms, r_i , are given. Equations (6) and (7) imply:

$$\hat{p}_i = \hat{a}_i + \frac{\delta s_i}{(\sigma - \delta s_i - 1)(\sigma - \delta s_i)} \hat{s}_i, \quad (29)$$

$$\hat{s}_i = \hat{r}_i - \sum_{j=1}^m s_j \hat{r}_j - (\sigma - 1) \left(\hat{p}_i - \sum_{j=1}^m s_j \hat{p}_j \right).$$

Substituting the last equation into (29) yields:

$$[1 + \beta_i(\sigma - 1)]\hat{p}_i - \beta_i(\sigma - 1) \sum_{j=1}^m s_j \hat{p}_j = \hat{a}_i + \beta_i \left(\hat{r}_i - \sum_{j=1}^m s_j \hat{r}_j \right), \text{ for all } i.$$

These equations can also be expressed as:

$$\mathbf{B}\hat{\mathbf{p}} = \mathbf{R}\hat{\mathbf{r}} + \hat{\mathbf{a}}, \quad (30)$$

where $\mathbf{B}$ is an $m \times m$ matrix with elements:

$$b_{ii} = 1 + \beta_i(\sigma - 1)(1 - s_i),$$

$$b_{ij} = -\beta_i(\sigma - 1)s_j, \text{ for } j \neq i,$$

$\hat{\mathbf{p}}$ is an $m \times 1$ column vector with elements p_i , where a hat represents a proportional rate of change (i.e., $\hat{p}_i = dp_i/p_i$), $\mathbf{R}$ is an $m \times m$ matrix with elements:

$$r_{ii} = \beta_i(1 - s_i),$$

$$r_{ij} = -\beta_i s_j, \text{ for } j \neq i,$$

$\hat{\mathbf{r}}$ is an $m \times 1$ column vector with elements $\hat{r}_i$, where a hat represents a proportional rate of change, and $\hat{\mathbf{a}}$ is an $m \times 1$ column vector with elements $\hat{a}_i$, where a hat represents a proportional rate of change.

Since

$$|b_{ii}| - \sum_{j \neq i} |b_{ij}| = 1 + \beta_i(\sigma - 1)(1 - \sum_{j=1}^m s_j) > 1,$$

$\mathbf{B}$ is a diagonally dominant matrix with positive diagonal and negative off-diagonal elements. It therefore is an M -matrix and its inverse has all positive entries. This inverse, denoted by $\tilde{\mathbf{B}} = \mathbf{B}^{-1}$, is therefore an $m \times m$ matrix with elements $\tilde{b}_{ij} > 0$. Next note that $\mathbf{B}$ can be expressed as:

$$\mathbf{B} = \mathbf{I} + (\sigma - 1)\mathbf{R},$$

where $\mathbf{I}$ is the identity matrix. Therefore:

$$\mathbf{B}^{-1}\mathbf{B} = \tilde{\mathbf{B}} + (\sigma - 1)\tilde{\mathbf{B}}\mathbf{R} = \mathbf{I}. \quad (31)$$

It follows from this equation that:

$$\begin{aligned} \tilde{b}_{ii} + (\sigma - 1) \sum_{j=1}^m \tilde{b}_{ij} r_{ji} &= 1, \\ \tilde{b}_{ik} + (\sigma - 1) \sum_{j=1}^m \tilde{b}_{ij} r_{jk} &= 0, \text{ for } k \neq i. \end{aligned}$$

Summing these up yields:

$$\sum_{k=1}^m \tilde{b}_{ik} + (\sigma - 1) \sum_{j=1}^m \tilde{b}_{ij} \sum_{k=1}^m r_{jk} = 1, \text{ for all } i. \quad (32)$$

Since:

$$\sum_{k=1}^m r_{jk} = \beta_j (1 - \sum_{k=1}^m s_k) > 0$$

and $\tilde{b}_{ik} > 0$ for all i and k , it follows from (32) that:

$$0 < \tilde{b}_{ik} < 1 \text{ for all } i \text{ and } k.$$

Equation (31) implies:

$$(\sigma - 1)\tilde{\mathbf{B}}\mathbf{R} = \mathbf{I} - \tilde{\mathbf{B}},$$

and therefore $\tilde{\mathbf{B}}\mathbf{R}$ has positive diagonal elements and negative off-diagonal elements.

Going back to the comparative statics equations (30), we have:

$$\hat{\mathbf{p}} = \tilde{\mathbf{B}}\mathbf{R}\hat{\mathbf{r}} + \tilde{\mathbf{B}}\hat{\mathbf{a}}.$$

It follows from the properties of $\tilde{\mathbf{B}}$ that a decline in a_i reduces every price p_j , but less than proportionately. Equation (29) then implies that all market share $s_j, j \neq i$, decline while the market share s_i rises. And it follows from the properties of $\tilde{\mathbf{B}}\mathbf{R}$ and (29) that an increase in r_i raises the price and market share of firm i and reduces the price and market share of every other firm $j \neq i$. Noting that the markup of every firm i is larger the larger its market share, we therefore have:

Proposition 9. *Suppose that the number of firms and their product range are given. Then: (i) an increase in r_i raises the price, markup and market share of firm i , and reduces the price, markup and market share of every other large firm; (ii) a decline in a_i reduces the price of every large firm less than proportionately, raises the markup and market share of firm i , and reduces the markup and market share of every other large firms.*

Aggregative Economy

In this section we show how to construct an aggregative economy with a continuum of industries, each one of the type analyzed in the main text of this paper.

We consider an economy with a continuum of individuals of mass 1, each one providing one unit of labor. The labor market is competitive and every individual earns the same wage rate.

There is a continuum of sectors of measure one, each one producing a differentiated product. Real consumption in sector k is:

$$X^k = \left[\int_0^{N^k} x^k(\omega)^{\frac{\sigma^k-1}{\sigma^k}} d\omega \right]^{\frac{\sigma^k}{\sigma^k-1}}, \quad \sigma^k > 1,$$

where N^k is the number of brands available in sector k , $x^k(\omega)$ is consumption of variety ω in sector k , and σ^k is the elasticity of substitution in sector k . Using this definition, the price index of X^k is:

$$P^k = \left[\int_0^{N^k} p^k(\omega)^{1-\sigma^k} d\omega \right]^{\frac{1}{1-\sigma^k}},$$

where $p^k(\omega)$ is the price of variety ω . The log utility of a representative individual is:

$$\log(u) = \int_0^1 \log(X^k) dk.$$

In these circumstances every individual spends an equal amount of money in every sector. Therefore, if E denotes aggregate spending per capita, spending per capita in sector k also equals E . In this event, aggregate demand for variety ω in sector k is:

$$x^k(\omega) = A^k p(\omega)^{-\sigma}, \quad (33)$$

$$A^k = E \left(P^k \right)^{\sigma^k - 1}. \quad (34)$$

An individual's inter-temporal utility function is:

$$U = \int_0^\infty e^{-\rho t} \log(u_t) dt,$$

where ρ is the subjective discount rate. As a result, the intertemporal allocation of spending satisfies:

$$\frac{\dot{E}_t}{E_t} = \zeta_t - \rho, \quad (35)$$

where ζ_t is the interest rate at time t .

Two types of firms operate in sector k : atomless single-product firms and large multi-product firms, each one with a positive measure of product lines. Single-product firms produce $\bar{r}^k > 0$ varieties, each one specializing in a single brand. Large firm i in sector k has $r_i^k > 0$ product lines, $i = 1, 2, \dots, m^k$, where m^k is the number of large firms in this sector. All the brands supplied to the market are distinct from each other.

All single-product firms share the same technology, which requires $\bar{a}^k$ unit of labor per unit output in sector k . Facing the demand function (33), a single-product firm maximizes profits $A^k p(\omega)^{-\sigma} [p(\omega) - \bar{a}^k]$, taking as given the demand shifter A^k . Therefore, a single-product firm prices its brand ω according to $p(\omega) = \bar{p}^k$, where:

$$\bar{p}^k = \frac{\sigma^k}{\sigma^k - 1} \bar{a}^k. \quad (36)$$

This yields the standard markup $\bar{\mu}^k = \sigma^k / (\sigma^k - 1)$ for a monopolistically competitive firm.

A large firm i has a technology that requires a_i^k units of labor per unit output, and it faces the demand function (33) for each one of its brands. As a result, it prices every brand equally. We denote this price by p_i^k . The firm chooses p_i^k to maximize profits $r_i^k A^k p_i^{-\sigma} (p_i - a_i^k)$. However,

unlike a single-product firm, a large firm does not view A^k as given, because it recognizes that

$$P^k = \left(\bar{r}^k (\bar{p}^k)^{1-\sigma} + \sum_{j=1}^{m^k} r_j^k (p_j^k)^{1-\sigma^k} \right)^{\frac{1}{1-\sigma^k}}, \quad (37)$$

and therefore that its pricing policy has a measurable impact on the price index of the differentiated product. It takes, however, the spending level E as given, because sector k is of measure zero. Accounting for this dependence of P^k on the firm's price, the profit maximizing price is:

$$p_i^k = \frac{\sigma^k - (\sigma^k - 1) s_i^k}{(\sigma^k - 1) (1 - s_i^k)} a_i^k, \quad (38)$$

where s_i^k is the market share of firm i in sector k and:

$$s_i^k = \frac{r_i^k (p_i^k)^{1-\sigma^k}}{(P^k)^{1-\sigma^k}} = \frac{r_i^k (p_i^k)^{1-\sigma^k}}{\bar{r}^k (\bar{p}^k)^{1-\sigma} + \sum_{j=1}^{m^k} r_j^k (p_j^k)^{1-\sigma^k}}. \quad (39)$$

Equations (38) and (39) jointly determine prices and market shares of large firms. The markup factor of firm i is $\mu_i^k = [\sigma^k - (\sigma^k - 1) s_i^k] / [(\sigma^k - 1) (1 - s_i^k)]$, which is increasing in its market share. When the market share equals zero the markup is $\sigma^k / (\sigma^k - 1)$, the same as the markup of a single product firm. The markup factor varies across firms as a result of differences in either the product span, r_i^k , or the marginal production cost, a_i^k . We analyze the dependence of prices, market shares and markups on marginal costs and product spans in the next section.

Entry of Single-Product Firms

The number of large firms in every sector, m^k , is given. Unlike large firms, however, single-product firms enter the industry until their profits equal zero. In every sector the firms play a two-stage game: in the first stage single-product firms enter; in the second stage all firms play a Bertrand game as described above. Under these circumstances, (36) and (38) portray the equilibrium prices, except that the number of single product firms, $\bar{r}^k$, is endogenous. We seek to characterize a subgame perfect equilibrium of this game.

To determine the equilibrium number of single-product firms, assume that they face an entry cost f^k in sector k and they enter until profits equal zero. In a subgame perfect equilibrium every entrant correctly forecasts aggregate spending on the sector's products, the number of entrants, and the price that will be charged for every variety in the second stage of the game. Therefore, every single-product firm correctly forecasts the price index and A^k . Using the optimal price (36) and the profit function $A^k p(\omega)^{-\sigma} [p(\omega) - \bar{a}^k]$, this free entry condition can be expressed as:

$$\frac{1}{\sigma^k} A^k \left(\frac{\sigma^k}{\sigma^k - 1} \bar{a}^k \right)^{1-\sigma^k} = f^k. \quad (40)$$

The left-hand side of this equation describes the operating profits, which equal a fraction $1/\sigma^k$ of revenue, while the right-hand side represents the entry cost. In these circumstances the demand shifter A^k is determined by f^k and $\bar{a}^k$, and it is rising in both f^k and $\bar{a}^k$. Importantly, it does not depend on the number of large firms nor on their product spans. Moreover, given the spending level E , which is determined at the economy-wide level and is not influenced by product spans in sector k (because the sector is of measure zero), the price index P^k is also independent of product spans in sector k . In particular, changes over time in this price index are driven by changes in aggregate spending. For this reason (34) and (35) imply:

$$\frac{\dot{P}_t^k}{P_t^k} = \frac{1}{\sigma^k - 1} (\rho - \zeta_t). \quad (41)$$

Optimal Control

We can now compute the response of p_i^k and s_i^k to changes in r_i^k as we did in the main text, and use the solution in the firm's optimal control problem. In the optimal control problem large firm i in sector k takes as given the path of the interest rate r_t and the path of spending E_t . After characterizing this solution we can use it to express the market clearing conditions. Spending E_t has to equal wage income and aggregate profits net of investment costs. This will give us the growth model. If we use the formulation from the main text, the steady state will have zero growth. But one could add a long-run growth mechanism, such as declining costs of innovation as a function of the cumulative experience in innovation, as is Romer (1990). The steady state should be easy to analyze in either case.

As in the main text, investment is given by

$$\dot{r}_i^k = \phi(\iota_i^k) - \theta r_i^k, \text{ for all } t \geq 0, \quad (42)$$

At every point in time the firms play a two stage game. In the first stage single-product firms enter and large firms invest in innovation. Single-product firms live only one instant of time. For this reason they make profits only in this single instant. Under the circumstances the demand shifter A^k is determined by the free entry condition, and it remains constant as long as the cost of entry and the cost of production of the single-product firms do no change. It follows that the profit flow of large firm i is:

$$\pi_i^k = r_i^k A^k \left(p_i^k \right)^{-\sigma} (p_i^k - a_i^k) - \iota_i^k, \text{ for all } t \geq 0,$$

where A^k is the same at every t while π_i^k , r_i^k , p_i^k and ι_i^k change over time, and p_i^k is given by $p_i^k = \frac{\sigma^k - (\sigma^k - 1)s_i^k}{(\sigma^k - 1)(1 - s_i^k)} a_i^k$. We can write the optimal control problem as:

$$\max_{\{\iota_i^k(t), r_i^k(t)\}_{t \geq 0}} \int_0^\infty e^{-\int_0^t \zeta_\tau d\tau} \pi_i^k \left[\iota_i^k(t), r_i^k(t) \right] dt$$

The main difference between this formulation and the formulation in the main text is that now we no longer have $\zeta_t = \rho$ at each point in time, but rather $\zeta_t = \frac{\dot{E}_t}{E_t} + \rho$. The current-value Hamiltonian of this problem is:

$$\mathcal{H}(\iota_i^k, r_i^k, \lambda_i^k) = \left\{ r_i^k A^k p_i^k(r_i)^{-\sigma} \left[p_i^k(r_i) - a_i^k \right] - \iota_i^k \right\} + \lambda_i^k \left[\phi(\iota_i^k) - \theta r_i^k \right],$$

and the first-order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \iota_i^k} &= -1 + \lambda_i^k \phi'(\iota_i^k) = 0, \\ -\frac{\partial \mathcal{H}}{\partial r_i^k} &= -\frac{\partial \left[r_i^k A^k (p_i^k)^{-\sigma} (p_i^k - a_i^k) \right]}{\partial r_i^k} + \theta \lambda_i^k = \dot{\lambda}_i^k - \zeta_t \lambda_i^k. \end{aligned}$$

Note that the path of the price index P_t^k is determined by the growth rate of the aggregate economy that each firm takes as exogenous. Therefore, the resulting first-order conditions have a similar form to those we derived in the main text:

$$\lambda_i^k \phi'(\iota_i^k) = 1, \quad (43)$$

$$\dot{\lambda}_i^k = (\zeta_t + \theta) \lambda_i^k - A^k p_i^k(r_i^k)^{-\sigma^k} \left\{ p_i^k(r_i^k) - a_i^k - r_i^k \left(\sigma^k p_i^k(r_i^k)^{-1} \left[p_i^k(r_i^k) - a_i^k \right] - 1 \right) \frac{dp_i^k(r_i^k)}{dr_i^k} \right\}. \quad (44)$$

Substituting (43) into (42) yields:

$$\dot{r}_i = \phi[\iota_i(\lambda_i)] - \theta r_i. \quad (45)$$

The second differential equation is obtained by substituting the pricing equation into (44):

$$\dot{\lambda}_i^k = (\zeta_t + \theta) \lambda_i^k - \Gamma_i^k(r_i^k), \quad (46)$$

where:

$$\Gamma_i^k(r_i^k) \equiv a_i^{1-\sigma^k} A^k \sigma \left[\frac{\sigma^k - (\sigma^k - 1) s_i^k(r_i^k)}{(\sigma^k - 1)(1 - s_i^k)} \right]^{-\sigma^k} \frac{1}{(\sigma^k - 1)(1 - s_i^k) \sigma + s_i(r_i)^2 (\sigma - 1)^2}. \quad (47)$$

Thus, our two differential equations are similar to the main text, with the caveat that the interest rate is evolving over time. Specifically, the dynamics are such that aggregate spending must satisfy

$$\zeta_t = \frac{\dot{E}_t}{E_t} + \rho.$$

In steady state:

$$\phi \left[\iota_i^k \left(\lambda_i^k \right) \right] = \theta r_i^k, \quad (48)$$

$$(\rho + \theta) \lambda_i^k = \Gamma_i^k \left(r_i^k \right), \quad (49)$$

where we have used the fact that in steady state $\zeta_t = \rho$. The comparative statics of this system have the same form as in the main text. But note that while the key condition for having an inverted-U relationship between productivity and product span was $(\sigma - \delta - 1)^2 (\sigma^2 - \delta^2) < (\sigma - 1) \delta^2$ in the main text, the formula is the same now with the exception that δ is replaced with $\sigma - 1$. This reduces the condition to $0 < (\sigma^k - 1)^3$, which is always satisfied. Thus, in this formulation we would expect every sector to have the inverted-U property. Another comparative static to note is the effect of an increase in the steady state expenditure level E . This shifts upward the curve associated with (49) in the phase diagram, resulting in an instantaneous increase in λ_i^k and a trajectory of further expansion of r_i^k and rising profits. Thus, firms growing in other sectors reinforce the market dominance of large firms across industries through a pecuniary externality.

In order to close the model we need to solve for the steady state expenditure level. The market clearing condition is simply that revenue must equal net profits plus the total wage bill. With a unit mass of labor and the wage rate as the numeraire, the resulting condition takes the form:

$$E_t = 1 + \int_{k \in K} \left[\sum_{i=1}^{m^k} r_i^k A^k \left(p_i^k \right)^{-\sigma} (p_i^k - a_i^k) - \iota_i^k \right] dk. \quad (50)$$

We can further simplify this by recalling that $A^k = E_t (P^k)^{\sigma^k - 1}$. This means that we can use (50) to obtain:

$$E_t = \left[1 - \int_{k \in K} \left[\sum_{i=1}^{m^k} r_i^k \left(P^k \right)^{\sigma^k - 1} \left(p_i^k \right) (p_i^k - a_i^k) - \iota_i^k \right] dk \right]^{-1}. \quad (51)$$

Thus, the steady state expenditure level is increasing in the net profits of large firms across sectors. This equation also holds at every point in time, noting that the optimal investment levels depend on the path of aggregate expenditure through the interest rate. It follows that in order to solve the path of spending we need to ensure that the paths of profits of all firms aggregates to the path that rationalizes the optimal investments at each point in time.