

Dynamics of Large Multinationals

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Abstract

We develop a model of large multinational enterprises, each one producing a continuum of products. These outsized firms compete as oligopolists in a domestic and foreign market, facing competitive pressure from monopolistically competitive single-product firms. The multinational enterprises engage in foreign direct investment (FDI) in order to expand the scope of products manufactured by their foreign affiliates, and they invest in research and development (R&D) in order to invent new products and thereby increase their overall product span. The paper focuses on two themes: substitutability between FDI and innovation and non-monotonicity in the time series and the cross section. We study the dynamic evolution of these enterprises and characterize transition dynamics and steady states. In addition to the evolution of product spans, we characterize the evolution of prices, markups, market shares, and exports relative to subsidiary sales. Furthermore, we study comparative dynamics that result from changes in trade costs, R&D costs, FDI costs and productivity changes of these firms. We also illustrate the use of our model for an analysis of mergers.

Keywords: multinational enterprise, foreign affiliate, foreign direct investment, innovation, firm dynamics, product span

JEL Classification: D43, F12, F23, L11, L13, L25

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1 Introduction

U.S. multinational enterprises (MNEs) play an oversized role in the world’s economy. According to the August 21, 2020 news release “Activities of U.S. Multinational Enterprises, 2018,” the Bureau of Economic Analysis (BEA) reports that worldwide employment of these companies was 43 million workers, with about one third employed abroad by majority-owned foreign affiliates, and their worldwide value-added was \$5.7 trillion. Furthermore, Antras and Yeaple (2014) report that only a quarter of goods sold to foreign customers by large American firms were domestically produced, while three quarters were produced by their foreign affiliates. About 90% of U.S. imports and exports flowed through these companies.

American multinationals play a big role in research and development (R&D) and they are relatively more active in R&D intensive sectors (see Antras and Yeaple (2014)). According to the BEA report, in 2018 American MNEs spent \$381.4 billion on R&D, with only \$58.2 billion accounted for by majority-owned foreign affiliates. More generally, within multinationals parents are relatively specialized in R&D while their subsidiaries are relatively specialized in serving foreign markets, and chiefly their host countries. Finally, both parents and subsidiaries tend to be larger, more productive, and more R&D intensive than non-multinational firms (see Antras and Yeaple (2014)).

Large firms, and multinational enterprises in particular, are multi-product firms (see Bernard *et al.* (2011); Hottman *et al.* (2016)). They serve foreign markets with multi-product exports and multi-product subsidiary sales. In a study of patterns of U.S. multinational corporations, Yeaple (2013) finds that “... many large U.S. firms simultaneously sell to unaffiliated customers in a given foreign market through exports from the United States and through locally based affiliates. While there are other possible interpretations of this fact, we feel that the most natural interpretation is that firms export a subset of their products and produce a different subset abroad.” (p. 2) In fact, Bernard *et al.* (2007) report that when firms export, they typically export more than one product. Moreover, “...firms that export more than one ten-digit Harmonized System product comprise 57.8 percent of exporting firms and account for more than 99.6 percent of export value.” (p. 119) Quite surprisingly, firms that export a single-product account for less than 0.4 percent of export value while firms that export more than five products account for 98 percent of export value. And as Yeaple points out, most of these large multi-product exporters are multinational corporations, whose exports amount to less than half their foreign sales.

Large, multi-product firms face competition from small single-product firms (see Hottman *et al.* (2016)). Small firms are typically short-lived while large firms are long-lived. Cao *et al.* (2019) report that 95% of U.S. firms had single establishments while their share in employment was 45%. During 1972-2007 an average of 72% of plants in manufacturing belonged to single-plant firms, but these firms produced only 22% of value added (see Table A1 in Kehrig and Vincent (2019)). These data suggest the existence of a competitive fringe faced by large multi-product MNEs.¹ These

¹Shimomura and Thisse (2012) study the interactions between a monopolistically competitive fringe of single-product firms and oligopolistic large firms in a static closed economy, while Parenti (2018) studies such interactions in a static open economy without MNEs.

studies also report that firm growth took place mostly through the extensive margin.

Our aim is to construct a model that incorporates these essential features, and analyze the dynamic evolution of an industry of this type. We are especially interested in studying foreign direct investment that expands the product span of foreign affiliates and R&D that expands the product scope of multinational enterprises. Our model features large multi-product firms that act as oligopolies in the domestic and foreign markets, while facing competitive pressure from short-lived single-product monopolistically competitive firms. As we show in subsequent sections, the model generates rich results. Yet, we emphasize two main themes. First, substitutability between an MNE’s overall product span and the product span of its foreign affiliates. In other words, whether expansion of a corporation’s overall product scope leads to more product lines in its foreign affiliates. Second, non-monotonicity that arises either in the time series or in the cross section. Here we have in mind whether the evolutions of key variables, such as prices or market shares, are similar in the short- and long-run in response to changes, such as a decline in the cost of FDI. And whether such variables are monotonically related to characteristics of firms, such as productivity.

There has been growing interest in studying innovation and multinational production. Most of this work has been designed for quantitative analysis, however, abstracting from firm dynamics. Arkolakis *et al.* (2018) provide a good example. Like us, they treat innovation as a process for creating new goods, but, unlike us, they assume that every firm specializes in a single product and its innovation consists of a one-time acquisition of this product. Otherwise, their quantitative model is static but rich in other dimensions; e.g., it has many countries and platform FDI. A firm that enters the industry draws a vector of productivities in which every entry applies to a different country and the analysis emphasizes relative specialization in innovation or production.

A quantitative model with similar features, but with firm dynamics, is developed in Garetto *et al.* (2021). Their firms draw time-invariant firm-specific productivities, which are subsequently augmented by time-varying country-specific shocks. The latter follow exogenous processes that feed the system’s dynamics. In this model, a firm that builds a foreign subsidiary acquires an option to sell in the host market as well as an option to export to other countries.²

In contrast to these papers, our model is stylized in its geographic structure. But this is compensated for by modeling multinational corporations as large multi-product firms with oligopolistic market power, as they indeed are. Moreover, we treat innovation and FDI as investment activities that gradually expand a corporation’s range of products and the span of products available to its foreign affiliates, respectively. Both investments involve sunk costs. As pointed out above, our goal is to study the dynamic evolution of such firms in a world economy in which they play an outsized role.³ These MNEs serve foreign markets with exports and subsidiary sales and they use market power to price to market, charging different prices at home and abroad.⁴

²Gumpert *et al.* (2020), a predecessor to Garetto *et al.* (2021), extends the static heterogeneous firms model of Helpman *et al.* (2004) by assuming that every firm’s productivity evolves according to a Markov process. They focus on the life-cycle of exporters and MNEs, using a simple spacial structure with no platform FDI.

³For this purpose we model them as oligopolists who face competition from a fringe of monopolistically competitive single-product firms, building on the closed-economy model of Helpman and Niswonger (2021).

⁴Helpman (1985) provides an early analysis of multi-product multinational corporations. In his paper, firms engage

Some basic features of our model are described in Section 2. Preferences are quasi-linear, there is one factor of production, labor, and two sectors: one producing a homogeneous good the other producing varieties of a differentiated product. As is common, wages are determined in the homogeneous sector. Individuals have CES preferences for varieties of the differentiated product. There are two countries, home and foreign, but only the former is home to large multi-product firms.

Section 3 describes an instantaneous equilibrium of the world economy that holds at each point in time. In every instant the firms play a two-stage game, taking as given the number of products owned by every multi-product firm and the number of products available to their foreign affiliates. First, single-product firms enter, expecting to survive one instant only. In the second stage, all firms play a Bertrand price game. For simplicity, we assume that single-product firms serve only their domestic markets. This simplification is close to the fact cited above, that single-product firms account for less than 0.4 percent of U.S. exports. In contrast, multi-product firms sell at home and abroad, and they can serve the foreign market with exports or foreign affiliates. Exports entail melting iceberg costs.

In Section 4, we analyze the evolution of FDI, assuming that every large firm has a constant product span. This simplified framework, which abstracts from product innovation, is designed to gain insights that are difficult to see in the more complex case of joint FDI and innovation. To begin with, in this case we can fully characterize the growth of MNEs, including transition dynamics and long-run outcomes. Second, it clearly identifies the mechanism that leads to substitutability between the product span of a firm and the product span of its foreign affiliates. An increase in the overall product span of a corporation raises its foreign market share, and an increase in the foreign market share reduces the value of a marginal product available to foreign affiliates. For this reason the firm reduces foreign direct investment and shrinks over time the product scope of its foreign subsidiaries. Third, it helps to unveil non-monotonicities in the time series. To illustrate, an increase in export costs reduces on impact foreign markups and market shares. But this one-time adjustment is followed by a subsequent gradual increase in foreign markups and market shares, because the initial fall in a market share raises the profitability of FDI and brings about a gradual expansion of the product scope of foreign subsidiaries. Despite that, exports relative to subsidiary sales grow throughout, in line with the proximity-concentration tradeoff theory of foreign direct investment.⁵ In contrast, an increase in FDI costs sets in motion monotonic evolutions of these variables. The product span of foreign affiliates gradually declines, as do foreign markups and market shares, while exports relative to subsidiary sales gradually increase.

These comparative dynamics are useful for thinking about Brexit, where rising trade and FDI

in monopolistic competition and choose product spans. But the model is static. Baldwin and Ottaviano (2001) also analyze a static model of multi-product multinational corporations. In their paper the product span is exogenous and the firms engage in reciprocal dumping.

⁵This theory emphasizes the tradeoff between variable trade costs and fixed costs of forming a foreign subsidiary. Its main prediction is that higher trade costs reduce the ratio of exports relative to subsidiary sales while higher fixed costs of foreign subsidiaries raise this ratio. See Markusen (1984) for the basic theory, Brainard (1997) for empirical evidence, and Helpman *et al.* (2004) for an extension to sectors with heterogeneous firms and supporting evidence. All these papers use a static framework.

costs have been identified as prominent features (see Dhingra and Sampson (2022)). The UK had a referendum in 2016 on leaving the European Union (EU), in which the majority of voters supported exit. Although details of the expected conditions of future trade and foreign direct investment costs with the EU were unknown at the time of the vote, it was widely perceived that both will increase. Brexit took place on January 31, 2020. Yet the economic relations between the UK and EU changed only in the beginning of 2021, with the signing of the Trade and Cooperation Agreement. This free trade agreement applies to tariffs and quotas, but it raises UK trade costs due to the introduction of customs and regulatory hurdles at the borders between the two economies. It also raises FDI costs, because it does not contain elements of deep integration that existed when the UK was a member of the Single Market, and it does not guarantee market access for services. Our model speaks to dynamic adjustments that may emerge from these types of rising costs.⁶

We also show in Section 4 that whenever the MNEs differ in labor productivity and the dispersion of productivity is large, the long-run cross-sectional relationship between product spans of foreign affiliates and labor productivity has an inverted-U shape. This finding illustrates potential non-monotonicities in cross-sectional relationships.

Our model can be used for various applications. For illustrative purposes, we apply it in Section 5 to a merger. We show that when two large corporations merge, the merged firm's combined market share is larger in the long-run than it would have been of each one of its component firms. Every foreign marginal product line is then less valued, FDI is lower and foreign affiliates have fewer product lines. The decline in the product scope of foreign subsidiaries is driven by the mechanism that brings about substitutability of the overall product span and the product span of foreign affiliates. The merged firm charges higher prices at home and abroad and the present value of its profits exceeds what would have been the present value of joint profits of its component firms. Higher profits benefit the home country. While potentially harmful to welfare, the higher prices charged by the merged corporation are offset by entry of single-product firms up to a point at which the price index of real consumption of the sector's products does not change. Consequently welfare rises.

The joint dynamics of FDI and innovation are analyzed in Section 6. In the case of a constant product scope of an MNE, we show that the dynamic path is globally saddle-path stable. With endogenous innovation, which generates two state variables—the firm's overall product span and the product span of its foreign affiliates—we show that the dynamic path is locally saddle-path stable around the steady state.

An analysis of the steady state displays in a sharper way substitutability between a corporation's product scope and the product scope of its foreign affiliates. Now a decline in FDI costs expands the product scope of foreign affiliates and *contracts* the overall product span of the firm. And a decline in innovation costs expands the overall product scope of the firm and *contracts* the product span of its foreign affiliates. Furthermore, the endogeneity of the overall product scope can lead to

⁶Reviewing studies of Brexit, Dhingra and Sampson (2022) point out that there has been little research on how it has affected foreign direct investment. Naturally, more will become known over time.

an inverted-U shape relationship between firm productivity levels and their overall product scopes as well as between firm productivity levels and their exports relative to subsidiary sales.

A summary of our main results is provided in Section 7.

2 Preliminaries

We consider a world consisting of two countries, a home country H and a foreign country F . In every country, there is a continuum of individuals of mass one. Labor markets are competitive and within a country, every individual earns the same wage rate.

There are two sectors. One sector produces a tradable homogeneous good with one unit of labor per unit output and this technology is available in both countries. Demand for the homogeneous good is always high enough to secure positive production in H and F . For this reason in every country the wage rate equals the price of the homogeneous good. We normalize this price to equal one and therefore wages equal one in both countries. The other sector produces varieties of a differentiated product.⁷ Fixed wages simplify the analysis and afford tractability to the more complex problems of market structure and firm dynamics in the differentiated product sector, which are the main issues addressed in this paper.

Every individual supplies a fixed amount of labor, l , and has a utility function⁸

$$u = x_0 + \frac{\varepsilon}{\varepsilon - 1} \left[\int_{\omega \in \Omega} x(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{(\varepsilon-1)\sigma}{\varepsilon(\sigma-1)}}, \quad \sigma > \varepsilon > 1, \quad (1)$$

where x_0 is consumption of the homogeneous good, $x(\omega)$ is consumption of variety ω of the differentiated product, Ω is the set of available varieties, σ is the elasticity of substitution between varieties and ε gauges the degree of substitutability between varieties of the differentiated product and the homogeneous good. The assumption $\sigma > \varepsilon$ asserts that brands of the differentiated product are better substitutes for each other than for the homogeneous good. The assumption $\varepsilon > 1$ ensures that aggregate spending on the differentiated product declines when its price rises (see below).

Real consumption of the differentiated product is

$$X = \left[\int_{\omega \in \Omega} x(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}}.$$

Consequently, the price index of X is

$$P = \left[\int_{\omega \in \Omega} p(\omega)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}},$$

where $p(\omega)$ is the price of variety ω .

Every individual chooses consumption to maximize utility subject to the budget constraint

⁷It is straightforward to generalize the analysis to multiple sectors with differentiated products, or to differences in labor productivity in the homogeneous sector that lead to wage differences across countries.

⁸We can allow l to vary across countries, but this variation will not affect our results.

$x_0 + PX = l + y$, where y is non-wage income that may differ across countries. This yields $X = P^{-\varepsilon}$ as long as consumers purchase the homogenous good and varieties of the differentiated product, which we assume always to be the case (this requires l to be large enough). In this case, the demand for variety ω is independent of y and equal to

$$x(\omega) = P^\delta p(\omega)^{-\sigma}, \delta = \sigma - \varepsilon > 0. \quad (2)$$

Aggregate spending on the differentiated product equals $PX = P^{1-\varepsilon}$, which declines in P (because $\varepsilon > 1$).⁹

In the foreign country, the differentiated product sector is populated by single-product firms that require f units of labor to enter the industry and a units of labor per unit output for manufacturing brands of the differentiated product. These firms serve only the domestic market.¹⁰ After entry, every single-product firm maximizes profits, $P_F^\delta p(\omega)^{-\sigma} [p(\omega) - a]$, taking as given the demand function (2) and the price index in country F , P_F . This results in the price

$$p = \frac{\sigma}{\sigma - 1} a \quad (3)$$

for domestic sales. We denote by $n_F > 0$ the number of single-product firms in country F , each one supplying a distinct brand of the product.

In country H , two types of firms operate in the differentiated product sector: atomless single-product firms and I large multi-product firms. Every large firm has a positive measure of product lines, n_i , $i = 1, 2, \dots, I$. As in country F , all single-product firms share the same technology, which requires f units of labor for entry and a units of labor per unit output.¹¹ They serve only the home country. After entry, a single-product firm maximizes profits, $P_H^\delta p(\omega)^{-\sigma} [p(\omega) - a]$, subject to the demand function (2), taking as given the price index in country H , P_H . As a result, similar to country F single-product firms, every single-product firm in H charges the price p , given by (3). We denote by n_H the number of brands produced by single-product firms in country H .

While country F has no large multi-product firms, multi-product firm i in country H has a technology for producing $n_i > 0$ varieties, each one with a_i units of labor per unit output. The assumption of identical technologies for varieties of a multi-product firm is an analytical simplification. Bernard *et al.* (2011) report heterogeneity of sales across products within multi-product firms, which suggests possible technological heterogeneity. Since this heterogeneity is not central to our main concerns, we assume that a_i can be used to produce each one of the n_i products at home or

⁹An individual's consumption choice yields the indirect utility function

$$v = l + y + \frac{1}{\varepsilon - 1} P^{1-\varepsilon},$$

where the third term on the right-hand side represents consumer surplus.

¹⁰Assuming that small firms serve only the domestic market simplifies the algebra. What's more, this assumption is consistent with the evidence; few single-product firms export and even fewer engage in foreign direct investment. We make below a similar assumption regarding small firms in country H . Recall from the Introduction that single-product firms account for less than 0.4% of U.S. exports.

¹¹We could allow the technologies of single-product firms to vary across countries without affecting our main results.

each one of $m_i < n_i$ products abroad, where m_i is the product span of the firm's foreign subsidiaries that have been acquired through foreign direct investments (FDI).¹² That is, every multi-product firm is a multinational enterprise (MNE) that also exports some of its varieties. In the U.S. data for 2000, more than 90% of exports were controlled by MNEs; see Bernard *et al.* (2009). Our formulation approximates this reality.

For now, we take both n_i and m_i as given, in order to examine momentary equilibria. In Section 4 we will introduce investment in foreign subsidiaries and study dynamics of FDI that determine the time pattern of m_i for a given product span n_i . We will also identify in that section the mechanism that generates substitutability between m_i and n_i . In Section 6 we will introduce investment in innovation that expands the overall number of products, n_i , and investigate the joint evolution of n_i and m_i , focusing on properties of the long-run equilibrium. This gradual buildup of the analysis, towards full-fledged dynamics of product innovation and foreign direct investment, is done for instructive reasons. In particular, the dynamic analysis of FDI with no product innovation in Section 4 provides valuable insights for understanding the joint dynamics of product innovation and FDI in Section 6.

Evidently, MNE i is not of measure zero and it does not take as given the price indexes P_J , $J = F, H$. Firm i serves the foreign market with m_i varieties via subsidiary sales and with $n_i - m_i$ varieties via exports. Exports are costly, and we denote with $\tau - 1 > 0$ the per unit export cost. That is, it takes τ units of a product to ship to the foreign country for one unit to arrive. We also use p_i to denote the price firm i charges for a product sold in the home market, $p_{e,i}$ for the price charged for an exported product in the foreign market, and $p_{m,i}$ for the price charged in the foreign market by the firm's foreign affiliates. Using this notation, the price indexes are

$$P_F = \left[n_F p^{1-\sigma} + \sum_{k=1}^I (n_k - m_k) p_{e,k}^{1-\sigma} + \sum_{k=1}^I m_k p_{m,k}^{1-\sigma} \right]^{\frac{1}{1-\sigma}}, \quad (4)$$

$$P_H = \left[n_H p^{1-\sigma} + \sum_{k=1}^I n_k p_k^{1-\sigma} \right]^{\frac{1}{1-\sigma}}. \quad (5)$$

At each point in time, the firms play a two-stage game. In the first stage, single-product firms enter. This yields n_F and n_H that are taken as given in stage two, and we assume that both are always positive. In stage two, every firm chooses its prices, taking as given pricing policies of its rivals. The subgame perfect equilibrium of this two stage game constitutes the instantaneous equilibrium at a point in time. It is characterized in the next section.

¹²Many recent quantitative models of multinational production introduce heterogenous productivities in different geographical locations, in order to handle platform FDI; see, for example, Garetto *et al.* (2021). Since we do not deal with platform FDI, we adopt the simplifying assumption of equal productivities at home and abroad.

3 Instantaneous Equilibrium

To study the instantaneous equilibrium, we begin with the second stage of the game, in which n_F and n_H are given (and so are n_i and m_i for $i = 1, 2, \dots, I$). At this stage the firms play a Bertrand game: every firm chooses its prices given the pricing policies of all other firms. This implies that all single product firms charge p to their domestic customers, given by (3).

Large multi-product firms recognize the impact of their pricing policy on the price indexes P_F and P_H . For this reason firm i , which seeks to maximize profits, solves the following problem:

$$\max_{p_i, p_{e,i}, p_{m,i}} n_i P_H^\delta p_i^{1-\sigma} (p_i - a_i) + (n_i - m_i) P_F^\delta p_{e,i}^{1-\sigma} (p_{e,i} - \tau a_i) + m_i P_F^\delta p_{m,i}^{1-\sigma} (p_{m,i} - a_i)$$

subject to (4)-(5). The solution to this problem yields (see appendix for details)

$$p_i = \frac{\sigma - \delta s_i}{\sigma - \delta s_i - 1} a_i, \quad (6)$$

$$p_{e,i} = \frac{\sigma - \delta s_{F,i}}{\sigma - \delta s_{F,i} - 1} \tau a_i, \quad (7)$$

$$p_{m,i} = \frac{\sigma - \delta s_{F,i}}{\sigma - \delta s_{F,i} - 1} a_i, \quad (8)$$

where s_i is the market share of firm i in H and $s_{F,i}$ is its combined market share in F , from exports and subsidiary sales, and

$$s_i = \frac{n_i p_i^{1-\sigma}}{P_H^{1-\sigma}} = \frac{n_i p_i^{1-\sigma}}{n_H p^{1-\sigma} + \sum_{k=1}^m n_k p_k^{1-\sigma}}, \quad (9)$$

$$\begin{aligned} s_{F,i} &= \frac{(n_i - m_i) p_{e,i}^{1-\sigma} + m_i p_{m,i}^{1-\sigma}}{P_F^{1-\sigma}} \\ &= \frac{(n_i - m_i) p_{e,i}^{1-\sigma} + m_i p_{m,i}^{1-\sigma}}{n_F p^{1-\sigma} + \sum_{k=1}^m (n_i - m_i) p_{e,k}^{1-\sigma} + \sum_{k=1}^m m_k p_{m,k}^{1-\sigma}}. \end{aligned} \quad (10)$$

Equations (6)-(10) jointly determine prices and market shares of large firms, given n_F , n_H and $\{n_i, m_i\}_{i=1}^I$. Note that (7) and (8) imply $p_{e,i} = \tau p_{m,i}$. Namely, every large firm charges for its exports a price that is τ times higher than the price for its subsidiary sales.

Recall from the price charged by single-product firms, p , that these firms' markup factor is $\sigma/(\sigma - 1)$ (see (3)). In contrast, the markup factors of firm i are $(\sigma - \delta s_i)/(\sigma - \delta s_i - 1)$ in the domestic market and $(\sigma - \delta s_{F,i})/(\sigma - \delta s_{F,i} - 1)$ in the foreign market. The latter applies to exports and subsidiary sales (see (6)-(8)); that is, the firm charges the same markup for all foreign sales. Moreover, the markups are increasing in the respective market shares; the markup in the domestic market is rising with the domestic market share while the foreign markups are rising with the foreign

market share. When a market share equals zero, the markup is $\sigma/(\sigma - 1)$, the same as the markup of a single-product firm. The domestic markups vary across firms as a result of differences in either the product span, n_i , or the marginal production cost, a_i , while the foreign markups vary across firms as a result of differences in n_i , a_i and the number of product lines of foreign affiliates, m_i .

U.S. barcode data supports our market structure, in which there are few large firms and a fringe of many very small firms. Hottman *et al.* (2016) report that around two-thirds of sales in every product group are generated by the ten largest firms, which account for 2% of the total number of firms. The remaining 98% of firms have market shares below 2%. Moreover, "...firms in the top decile of sales are multiproduct firms, supplying on average 68 different goods with the largest firms supplying hundreds of goods." (p. 1302) They also show that the markups of the three largest firms in a product group are large, yet decline rapidly with firm size. Our model is consistent with these findings.

The system of equations (6)-(10) is separable. Prices and market shares of sales in H can be solved with equations (6) and (9) only, while prices and market shares of sales in country F can be solved with equations (7), (8) and (10) only. Firm i 's exports relative to subsidiary sales are given by

$$\rho_i = \tau^{1-\sigma} \frac{n_i - m_i}{m_i}. \quad (11)$$

These measures are central to the proximity-concentration tradeoff theory of foreign direct investment.¹³ According to this theory, a firm can serve a foreign market with exports or subsidiary sales. Exporting entails variable export costs while subsidiary sales require foreign direct investment (a fixed cost). Therefore the tradeoff is between larger fixed costs of FDI or larger variable costs of exporting. High FDI costs tilt the balance in favor of exporting while high trade costs tilt the balance in favor of foreign investment (see Markusen (1984) for the basic theory, Brainard (1997) for empirical evidence and Helpman *et al.* (2004) for an extension of the theory to sectors with heterogeneous firms and for empirical evidence). We will examine the consistency of our model with this theory in the next sections.

We now turn to the first stage of the game. Unlike large multi-product firms, single-product firms enter the industry until their profits equal zero. They are short lived. In particular, they live a single instant. A single-product firm correctly forecasts the price indexes P_F and P_H that will exist in the second stage of the game, and it understands that it cannot affect them. It also correctly forecasts the price it will charge for its product in the second stage of the game, given by (3). For

¹³Aggregate exports relative to aggregate subsidiary sales are given by

$$\rho = \tau^{1-\sigma} \frac{\sum_{i=1}^I (n_i - m_i) p_{m,i}^{1-\sigma}}{\sum_{i=1}^I m_i p_{m,i}^{1-\sigma}} = \sum_{i=1}^I \kappa_i \rho_i,$$

where

$$\kappa_i = \frac{m_i p_{m,i}^{1-\sigma}}{\sum_{k=1}^I m_k p_{m,k}^{1-\sigma}}.$$

That is, aggregate exports relative to subsidiary sales are a weighted average of exports relative to subsidiary sales of the multinational firms.

these reasons, free entry leads to operating profits that equal entry cost. For single-product firms in countries H and F this implies

$$\frac{1}{\sigma} P_J^\delta \left(\frac{\sigma}{\sigma-1} a \right)^{1-\sigma} = f, \quad J = H, F,$$

where the left-hand side describes the operating profits from sales in country J , equal to fraction $1/\sigma$ of revenue, while the right-hand side represents entry costs. Taken together, these free entry conditions imply that $P_F = P_H = P$, where P satisfies

$$\frac{1}{\sigma} P^\delta \left(\frac{\sigma}{\sigma-1} a \right)^{1-\sigma} = f. \quad (12)$$

In these circumstances, P is determined by the demand parameters σ and ε and the technological parameters of single-product firms f and a . Importantly, P does not depend on the number of large firms, their product spans nor the spans of their subsidiaries.

Our assumption of an infinitely fast turnover of single-product firms (who live one instant only) approximates the empirical evidence—cited in the Introduction—that the rate of turnover of small firms is much faster than the rate of turnover of large firms (see Cao *et al.* (2019)). This specification anchors the price indexes, thereby greatly simplifying the dynamic analysis in the following sections.¹⁴

Since the price index P is constant and the same in both countries, (6) and (9) provide solutions to the domestic price and market share in country H , p_i and s_i , respectively, as functions of the product span n_i . We denote these functions by $p_i(n_i)$ and $s_i(n_i)$ and note that they are increasing in n_i . They also depend on the firm's labor productivity $1/a_i$ and the price index P , although for now we emphasize their dependence on the product span only. Moreover, (8) and (10) together with $p_{e,i} = \tau p_{m,i}$ provide solutions to the foreign price and foreign market share as functions of n_i and m_i , which we denote by $p_{m,i}(n_i, m_i)$ and $s_{F,i}(n_i, m_i)$, respectively. These functions also depend on $1/a_i$ and P . Properties of these functions are elaborated in the appendix. We show that

$$p_{m,i}(n_i, m_i) \equiv p_i \left[(n_i - m_i) \tau^{1-\sigma} + m_i \right], \quad (13)$$

$$s_{F,i}(n_i, m_i) \equiv s_i \left[(n_i - m_i) \tau^{1-\sigma} + m_i \right], \quad (14)$$

where $p_i[\cdot]$ and $s_i[\cdot]$ are the domestic price and market share functions. We interpret $n_{F,i} := (n_i - m_i) \tau^{1-\sigma} + m_i$ as the *effective* number of products in the foreign market, where the number of exported goods is discounted by the trade cost statistic $\tau^{1-\sigma}$. In the absence of trade costs, or when the firm's foreign subsidiaries manufacture its entire product span, i.e., $n_{F,i} = n_i$, the firm charges at home the same price that its subsidiaries charge abroad, its market shares are the same in both countries and so are its markups. Otherwise the effective number of products in the foreign

¹⁴We show in the appendix of Helpman and Niswonger (2021) that without binding the price indexes in this way, the dynamics of large multi-product firms become extremely complicated, even in a simpler model of a closed economy.

country is smaller than n_i , as we assume. We therefore have

Proposition 1. *Suppose that $m_i \in (0, n_i)$. Then the price charged by MNE i 's foreign affiliates is lower than the domestic price, $p_{m,i} < p_i$, the foreign market share is smaller than the domestic market share, $s_{F,i} < s_i$, and markups of exporter and foreign affiliate sales are smaller than domestic markups.*

One may conclude from this proposition that large firms discriminate between domestic and foreign markets, because—despite facing comparable cost and demand structures—foreign subsidiaries charge lower prices than their parents at home. And indeed they do. The difference in treatment stems from the fact that only a fraction of a firm's products are manufactured abroad while the remaining fraction is exported. Despite the presence of export costs, the firm's best strategy is to maintain parity between its markups on exports and subsidiary sales and keep them lower than the domestic markup. As a result, the foreign market share is lower than the domestic market share.

4 FDI Dynamics

This section addresses the dynamics of foreign direct investment for a given overall product span, before presenting the full dynamic analysis of FDI and innovation in Section 6. Focusing the analysis on foreign direct investment, enables us to characterize transition dynamics in addition to steady state outcomes. Insights gained from this analysis help understanding the more complex interactions between innovation and investments in foreign subsidiaries. They also help understanding two key themes of this paper: (i) substitutability between a firm's overall product span and the product span of its foreign affiliates; and (ii) non-monotonic responses to economic shocks and non-monotonic relations between key variables in the cross-section of firms.

As stated above, we now examine the dynamics of foreign direct investment of a large firm i that has a fixed product span, n_i , where FDI affects the evolution of the foreign product span, m_i . This analysis is greatly aided by the free entry conditions of single-product firms, which secure a constant price index P on the dynamic path (see (12)). To this end, we assume that small single-product firms enter the differentiated product sector in both countries at each point in time, i.e., $n_F > 0$ and $n_H > 0$ on an equilibrium path.

Assume that firm i has a costly investment technology that expands the number of product lines of its subsidiaries, and that a fraction θ of m_i is destroyed per unit time.¹⁵ In particular, we assume that the change in the product span of subsidiaries follows the differential equation

$$\dot{m}_i = \phi_m(\iota_{m,i}) - \theta m_i, \quad (15)$$

where $\iota_{m,i}$ is foreign direct investment and $\phi_m(\iota_{m,i})$ is gross addition to the firm's foreign product

¹⁵We do not take a position on how many subsidiaries the firm has in the foreign country, because for our purposes only the size of m_i matters. For example, one can think about m_i as the number of plants owned by firm i , each one producing a different variety and each one belonging to a different subsidiary, or an organizational form in which all plants are owned by a single subsidiary

span per unit time.¹⁶ All the variables in this equation are time dependent, although we have suppressed the time index for simplicity. We assume that the function $\phi_m(\iota)$ is increasing, concave, $\phi_m(0) = 0$ and it satisfies the Inada conditions $\lim_{\iota \rightarrow 0} \phi'_m(\iota) = +\infty$ and $\lim_{\iota \rightarrow +\infty} \phi'_m(\iota) = 0$.

Next assume that the interest rate is constant and equal to r .¹⁷ Firm i maximizes the discounted present value of profits. However, as long as n_i is constant there is no dynamic decision regarding the present value of profits from domestic sales. For a constant P , equations (6) and (9) provide solutions to the domestic price and market share that depend on the firm's product span, $p_i(n_i)$ and $s_i(n_i)$, respectively.¹⁸ Consequently, domestic prices and market shares of large firms do not change over time, and neither do their domestic markups. Under these circumstances the only dynamic problem of a large firm is to maximize the present value of profits from foreign sales net of FDI costs, for a given value of n_i . For that reason firm i solves the following optimal control problem:

$$\max_{\{\iota_{m,i}(t), m_i(t)\}_{t \geq 0}} \int_0^\infty e^{-rt} \pi_{F,i}[\iota_{m,i}(t), m_i(t)] dt$$

subject to (15),

$$\pi_{F,i}(\iota_{m,i}, m_i) := P^\delta [(n_i - m_i) \tau^{1-\sigma} + m_i] p_{m,i}(n_i, m_i)^{-\sigma} [p_{m,i}(n_i, m_i) - a_i] - \iota_{m,i}, \text{ for all } t \geq 0, \quad (16)$$

the initial condition $m_i(0) = m_i^0$ and a transversality condition to be described below. In this problem $\iota_{m,i}$ is a control variable while m_i is a state variable.

The current-value Hamiltonian of this problem is

$$\begin{aligned} \mathcal{H}(\iota_{m,i}, m_i, \zeta_{m,i}) &= P^\delta [(n_i - m_i) \tau^{1-\sigma} + m_i] p_{m,i}(n_i, m_i)^{-\sigma} [p_{m,i}(n_i, m_i) - a_i] \\ &\quad + \zeta_{m,i} [\phi_m(\iota_{m,i}) - \theta m_i] - \iota_{m,i}, \end{aligned}$$

where $\zeta_{m,i}$ is the co-state variable of (15). Since this Hamiltonian is concave in $\iota_{m,i}$ and m_i (see below), a dynamic path that satisfies the first-order conditions and the transversality condition maximizes the present value of foreign profits net of FDI costs. Assuming an interior solution, such that $m_i \in (0, n_i)$ at each point in time, the first-order conditions of this optimal control problem are

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \iota_{m,i}} &= \zeta_{m,i} \phi'_m(\iota_{m,i}) - 1 = 0, \\ -\frac{\partial \mathcal{H}}{\partial m_i} &= -\frac{\partial \pi_{F,i}}{\partial m_i} + \zeta_{m,i} \theta = \dot{\zeta}_{m,i} - r \zeta_{m,i}, \end{aligned}$$

¹⁶The variable $\iota_{m,i}$ does not correspond to the size of FDI in balance of payments statistics. The latter represents financial flows while our variable represents real investment in new product lines, independently of financing sources.

¹⁷A constant interest rate can be derived from the assumption that consumers maximize the expected present value of utility subject to their budget constraints. Due to the quasi-linearity of the utility function, this yields an interest rate equal to the consumers' subjective discount rate.

¹⁸These functions also depend on a_i and P ; see appendix for their properties.

and the transversality condition is

$$\lim_{t \rightarrow \infty} e^{-rt} \zeta_{m,i}(t) m_i(t) = 0.$$

In addition, the optimal path has to satisfy the differential equation (15), starting at $m_i(0) = m_i^0$.

We show in the appendix that

$$\frac{\partial \pi_{F,i}}{\partial m_i} = (1 - \tau^{1-\sigma}) \Lambda_i [s_{F,i}(n_i, m_i)],$$

where

$$\Lambda_i(s) := \left(\frac{\sigma - \delta s}{\sigma - \delta s - 1} \right)^{-\sigma} \frac{a_i^{1-\sigma} P^\delta \sigma}{(\sigma - \delta s - 1) \sigma + (s\delta)^2}$$

is a decreasing function, i.e., $\Lambda_i'(s) < 0$. Clearly, $(1 - \tau^{1-\sigma}) \Lambda_i [s_{F,i}(n_i, m_i)]$ represents marginal profits of the product span of foreign subsidiaries. It is declining in the foreign market share and rising in the variable export cost τ . As is evident from (14), however, the foreign market share also depends on export costs. For this reason, the total impact of τ on marginal profitability of m_i combines the direct and indirect effects. Since both are positive, we conclude that marginal profits of m_i are rising in τ . Furthermore, since (14) implies that the foreign market share is increasing in n_i and m_i , we obtain¹⁹

Lemma 1. Marginal profits from exports plus subsidiary sales that result from an increase in m_i , are

$$\frac{\partial \pi_{F,i}}{\partial m_i} = (1 - \tau^{1-\sigma}) \Lambda_i \{s_i [(n_i - m_i) \tau^{1-\sigma} + m_i]\}.$$

They are increasing in τ and decreasing in m_i and n_i .

To understand this Lemma, note that firm i serves the foreign market with both exports and subsidiary sales. For this reason the profitability of each form of sales depends on the other and their joint profits, $\pi_{F,i}$, depend on the composition of exports vs. subsidiary sales. With a constant number of products, n_i , an expansion of m_i spells a replacement of exported varieties with varieties sold by foreign subsidiaries. This raises joint profits, because unit export costs are higher than unit subsidiary sales costs. Hence, $\partial \pi_{F,i} / \partial m_i > 0$. Intuitively, the marginal profitability gain is larger the larger this unit cost differential, and the cost differential is increasing in τ . Also intuitively, this marginal profitability gain is smaller the larger the firm's market share in country F , and this market share is rising with the *effective* number of products, $n_{F,i} = (n_i - m_i) \tau^{1-\sigma} + m_i$ (the physical number of products is n_i). Since $n_{F,i}$ is increasing in the firm's overall product span, marginal profit gains from expanding the product lines of foreign affiliates declines with n_i . Furthermore, an increase in m_i also reduces this marginal profit gain, because—due to export costs—a replacement of an exported variety with a variety supplied by a foreign subsidiary raises $n_{F,i}$, and consequently raises

¹⁹This lemma shows that the Hamiltonian is concave in m_i . It is also concave in $\iota_{m,i}$ due to concavity of the FDI function $\phi_m(\iota_{m,i})$.

the foreign market share. As a result, marginal profits from replacing an exported product with a product supplied by foreign subsidiaries decline.

The finding that an increase in the overall span of available products reduces marginal profitability of investment in foreign affiliates, signals that FDI substitutes for investment in product innovation, which is the first main theme of this paper. As a firm expands the number of product lines via innovation, it finds it less profitable to invest in foreign affiliates; a dynamic link that we examine in Section 6.

The above first-order conditions of the optimal control problem can be expressed as:

$$\begin{aligned}\zeta_{m,i}\phi'_m(\iota_{m,i}) &= 1, \\ \dot{\zeta}_{m,i} &= (r + \theta)\zeta_{m,i} - (1 - \tau^{1-\sigma})\Lambda_i\{s_i[(n_i - m_i)\tau^{1-\sigma} + m_i]\}.\end{aligned}\quad (17)$$

These conditions have an intuitive interpretation, once it is recognized that $\zeta_{m,i}$ measures the value of a marginal product line in the foreign subsidiaries. The first condition states that investing one dollar in adding foreign product lines (the right-hand side) equals the value of the product lines created by this investment (the left-hand side). The expansion of product lines is given by $\phi'_m(\iota_{m,i})$, and every product line is valued at $\zeta_{m,i}$. Next, equation (17) represents a standard no-arbitrage condition for asset pricing, where the asset is a product line. It states that the asset's flow of earnings plus expected capital gains equal interest income. Earnings are

$$\frac{\partial \pi_{F,i}}{\partial m_i} = (1 - \tau^{1-\sigma})\Lambda_i\{s_i[(n_i - m_i)\tau^{1-\sigma} + m_i]\}.$$

Expected capital gains consist of two parts: the rise in the value of the asset $\dot{\zeta}_{m,i}$, and a total capital loss $\zeta_{m,i}$ that occurs with probability θ . Accordingly, the expected capital gain is $\dot{\zeta}_{m,i} - \theta\zeta_{m,i}$. Finally, interest earnings equal $r\zeta_{m,i}$. Taken together, these pieces deliver (17), which states that investing in an interest-bearing asset yields the same rate of return as investing in a new product line in the foreign country.

Using $\zeta_{m,i}\phi'_m(\iota_{m,i}) = 1$ to solve the FDI flow,

$$\iota_{m,i} = \varphi_m\left(\zeta_{m,i}^{-1}\right),$$

where $\varphi_m(\cdot)$ is the inverse of $\phi'_m(\cdot)$, and substituting the result into (15) yields the differential equation

$$\dot{m}_i = \phi_m\left[\varphi_m\left(\zeta_{m,i}^{-1}\right)\right] - \theta m_i. \quad (18)$$

This equation describes the dynamics of the number of foreign product lines. Given the value of a new product line, $\zeta_{m,i}$, firm i chooses foreign direct investment up to the point at which the value of the new foreign product lines just equals FDI costs. This level of investment, $\varphi_m\left(\zeta_{m,i}^{-1}\right)$, adds $\phi_m\left[\varphi_m\left(\zeta_{m,i}^{-1}\right)\right]$ new product lines. Since a fraction θ of the product lines becomes obsolete, the net expansion of the product scope of the firm's foreign affiliates follows (18).

Equations (17) and (18) represent an autonomous system of two differential equations with one initial condition, $m_i(0) = m_i^0$, and a free choice of $\zeta_{m,i}(0)$. They yield the steady state requirements

$$(r + \theta) \zeta_{m,i} = (1 - \tau^{1-\sigma}) \Lambda_i \{s_i [(n_i - m_i) \tau^{1-\sigma} + m_i]\}, \quad (19)$$

$$\phi_m \left[\varphi_m \left(\zeta_{m,i}^{-1} \right) \right] = \theta m_i, \quad (20)$$

obtained by imposing $\dot{m}_i = \dot{\zeta}_{m,i} = 0$. Note that in the steady state, the value of a marginal product line, $\zeta_{m,i}$, represents the discounted present value of marginal profits from the foreign product span, where future profits are discounted with the interest rate, r , plus the rate of depreciation, θ . The discounted present value of marginal profits induces in turn an FDI level that compensates for the attrition of the foreign product span, so that m_i does not change.

The transversality condition is satisfied when the dynamic path converges to a steady state and Figure 1 describes such an equilibrium path, where curve $\dot{\zeta}_{m,i} = 0$ satisfies (19) and curve $\dot{m}_i = 0$ satisfies (20). This path is saddle-path stable. Starting with an initial m_i^0 , the co-state variable $\zeta_{m,i}$ acquires at time zero a value that positions the differential equations on the saddle path. When $m_i(0)$ is below its steady state value, m_i rises over time and $\zeta_{m,i}$ declines. In the opposite case, when $m_i(0)$ is above its steady state value, m_i declines over time and $\zeta_{m,i}$ rises. Since the foreign market share—that combines exports and subsidiary sales—is an increasing function of the foreign product span, these dynamics imply that the foreign market share and foreign markup rise when m_i increases and decline when m_i declines. In contrast, the domestic market share and markup do not change. Moreover, (11) implies that exports relative to subsidiary sales decline when m_i rises and rise when m_i declines. We therefore have

Proposition 2. *Suppose that n_i is constant. Then: (i) if $m_i(0)$ is below its steady state value, m_i rises during the transition and so do the foreign price, markup and market share, while exports relative to subsidiary sales decline; (ii) if $m_i(0)$ is above its steady value, m_i declines during the transition and so do the foreign price, markup and market share, while exports relative to subsidiary sales rise; (iii) there are no changes in the domestic price, markup and market share.*

This proposition shows that whenever a large multi-product multinational company expands its foreign operations via foreign direct investment (because the firm's foreign product span is below its long-run value), it chooses to raise prices to foreign consumers from both exports and subsidiary sales. Moreover, in the process it substitutes exports for subsidiary sales at the extensive margin, and thereby reduces the ratio of exports to subsidiary sales. This is an interesting dynamic implication of the proximity-concentration tradeoff theory of foreign direct investment.

There are potential changes in the number of single-product firms during the transition to a steady state. Since this adjustment process has no impact on the market in country H , the number of single-product firms does not change in that country. We show in the appendix, however, that a multinational enterprise that expands its foreign product span brings about a gradual decline in the number of single-product firms in the foreign country, with the opposite taking place when the firm

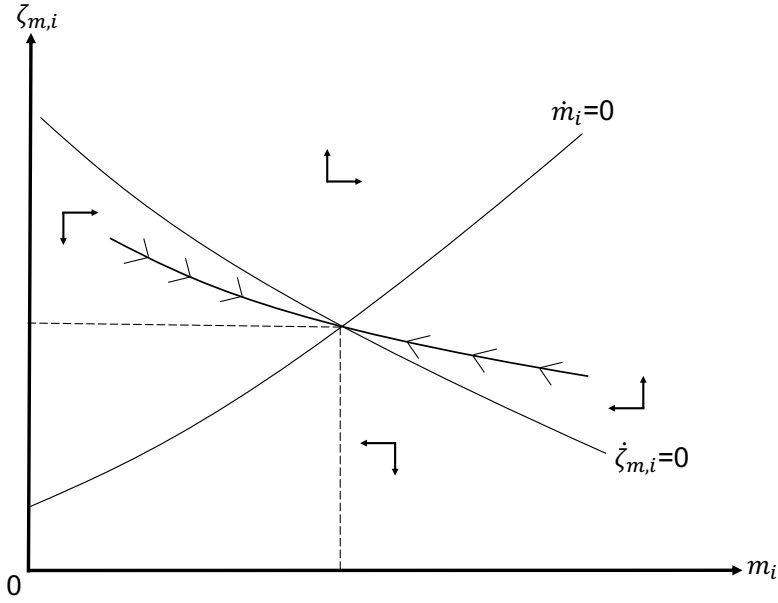


Figure 1: Transition Dynamics

contracts its foreign product span. Intuitively, expansion of a multinational's foreign operations raises its foreign market share, which reduces profitability of single-product firms. This fall in profitability discourages entry, and reduces the number of single-product firms. Reduced entry proceeds until profits from entry equal entry costs. We summarize these findings in

Corollary 1. *Suppose that n_i is constant. Then the home country's number of single-product firms does not change during the transition to a steady state. In addition, if $m_i(0)$ is below its steady state value, the number of foreign single-product firms declines over time, and if $m_i(0)$ is above its steady state value, the number of foreign single-product firms rises over time.*

4.1 Comparative Dynamics

In this section, we study the response of foreign direct investment and other variables of interest to changes in the economic environment. We begin with export costs and costs of foreign direct investment. Brexit provides motivation for this theoretical analysis, as explained in the Introduction.²⁰

Consider an increase in τ , starting from a steady state. It shifts upward the $\dot{\zeta}_{m,i} = 0$ curve (because the right-hand side of (19) is rising in τ) and does not alter the $\dot{m}_i = 0$ curve. In response, $\zeta_{m,i}$ jumps upward on impact to the new transition path on which m_i rises and $\zeta_{m,i}$ declines over time. In the new steady state foreign subsidiaries manufacture a larger set of products and fewer products are exported from the home country.

Exports relative to subsidiary sales decline on impact (see (11)) and the firm gradually exports

²⁰While we use Brexit to motivate the interest in rising trade and FDI costs, we do not claim to provide an analysis of Brexit per se.

less relative to subsidiary sales during the transition, in line with the proximity-concentration trade-off theory of foreign direct investment.²¹ Moreover, the instantaneous fall in exports reduces the firm's foreign market share, $s_{F,i}$, but this market share gradually rises thereafter as foreign investment broadens the foreign product span.²² Since foreign markups (on exports and subsidiary sales) are increasing in the foreign market share, they follow the same pattern as the foreign market share, and so does the price charged by foreign affiliates. In contrast, the export price rises on impact despite the fall in the foreign market share, because the affect of the market share is smaller than the direct effect of τ on export costs.²³ For this reason the export price rises on impact and continues to increase on the transition path, as the foreign market share rises.

Intuitively, a rise in export costs raises the value of a marginal product line in foreign subsidiaries. Why? Because it reduces export profitability and brings about a decline of export volumes. As a result, the firm's foreign market share declines on impact, and the value of a marginal foreign product line rises. Hence the upward jump in $\zeta_{m,i}$ on impact. But the instantaneous rise in the value of a marginal foreign product line makes it more profitable to engage in foreign direct investment, which gradually leads to an expansion of the product span of foreign subsidiaries. Therefore, exports decline relative to subsidiary sales on impact, and they further decline during the transition to the new steady state.

These findings are summarized in

Proposition 3. *Suppose that n_i is constant for $i = 1, 2, \dots, I$, and every MNE is in a steady state. Then MNE i responds to an increase in export costs by (i) raising on impact export prices and reducing prices charged by foreign affiliates, foreign markups, the foreign market share, and the firm's exports relative to subsidiary sales; (ii) subsequently gradually increasing the foreign product span, all foreign prices, foreign markups, the foreign market share, and reducing exports relative to subsidiary sales.*

This Proposition identifies a set of non-monotonic responses, which is the second main theme of this paper. First, the value of a foreign marginal product line rises on impact and gradually declines thereafter. Second, the foreign market share declines on impact and gradually increases thereafter. Finally, markups and prices of foreign affiliates decline on impact and gradually increase thereafter. Also of note is the result that export prices and prices of foreign affiliate move in opposite direction on impact, yet all rise subsequently, during the transition.

²¹Recall that firm i 's exports relative to subsidiary sales equal (see (11))

$$\rho_i = \tau^{1-\sigma} \frac{n_i - m_i}{m_i}.$$

Therefore every large firm reduces this ratio on impact in response to an increase in τ and this ratio further declines as m_i rises. If the foreign prices $p_{m,i}$, $i = 1, 2, \dots, I$ were constant, this would imply that aggregate exports relative to subsidiary sales, ρ , decline (see footnote 13). But these prices decline on impact for all firms, as is evident from (13), and rise gradually afterwards. They therefore exert an independent effect on ρ . For this reason the impact on ρ is ambiguous.

²²This can be seen from the dynamics of the effective number of products, $n_{F,i} = (n_i - m_i) \tau^{1-\sigma} + m_i$, by recalling that the foreign market share is increasing in $n_{F,i}$. The rise in τ reduces $n_{F,i}$ on impact, while the subsequent gradual increase in m_i raises it.

²³Note from (35) in the appendix that $\hat{p}_{e,i} = \hat{p}_{m,i} + \hat{\tau} > 0$.

Next, consider a productivity improvement that reduces FDI costs. This can be represented by an upward shift in a variable z_m , starting with $z_m = 1$, where we replace the function $\phi_m(\iota)$ with $z_m\phi_m(\iota)$. This productivity improvement has no impact on the $\dot{\zeta}_{m,i} = 0$ curve, but it shifts rightward the $\dot{m}_i = 0$ curve.²⁴ As a result, $\zeta_{m,i}$ jumps down on impact and sets off a dynamic trajectory on which m_i rises and $\zeta_{m,i}$ declines over time.

Interestingly, the decline in FDI costs reduces on impact the value of foreign product lines, but nevertheless leads to higher investment in foreign affiliates that gradually enlarges their product span. Foreign prices and markups do not change on impact, and neither does the foreign market share. However, they gradually rise during the transition in response to the growth of foreign product lines. Similarly, exports relative to subsidiary sales do not change on impact, but the MNE gradually reduces this ratio as the number of foreign product lines expands.

We summarize these findings in²⁵

Proposition 4. *Suppose that n_i is constant for $i = 1, 2, \dots, I$, and every MNE is in a steady state. Then MNE i responds to a productivity improvement in FDI by gradually increasing its foreign product span, all foreign prices, foreign markups, the foreign market share, and by reducing exports relative to subsidiary sales.*

Using the last two Propositions, we can think about Brexit as a combination of an increase in trade costs, τ , and an increase in FDI costs, i.e., a decline in z_m . Our model predicts in this case a short run rise in UK's export prices to the EU and a decline in prices of British affiliates in the EU. It also predicts a short-run decline in British markups and market share in the EU, as well as a fall in exports relative to subsidiary sales. But the model has no sharp prediction about the evolution of the product span of foreign affiliates, because higher trade costs encourage its expansion while higher FDI costs encourage its contraction. Be that as it may, the model predicts future falling prices of exports and subsidiary sales, falling markups and a falling market share, if the product spans of British affiliates in the EU will decline over time. The size of the hike in trade costs relative to the hike in FDI costs will determine these dynamics.

We next examine a one-time increase in the total number of products, n_i . This analysis helps understanding whether innovation and FDI are substitutes or complements, which is one of the two main themes of this paper. An increase in n_i shifts downward curve $\dot{\zeta}_{m,i} = 0$ in Figure 1 and does not alter the $\dot{m}_i = 0$ curve (see the steady state conditions (19) and (20)). In response, the value of a foreign product line, $\zeta_{m,i}$, jumps down on impact to a new transition path, on which it gradually rises while m_i gradually declines. These dynamics imply that the firm's foreign market share rises

²⁴In the presence of the productivity shifter z_m the steady state condition (20) is replaced with

$$z_m\phi_m[\varphi_m(1/z_m\zeta_{m,i})] = \theta m_i.$$

Since the left-hand side of this equation is rising in z_m , an increase in z_m shifts rightward the $\dot{m}_i = 0$ curve.

²⁵Using arguments similar to those in the proof of Corollary 1, it is straightforward to see that the response of the number of foreign single-product firms is opposite to the response of foreign prices. E.g., the number of foreign single-product firms rises on impact and declines subsequently in response to an increase in export costs. For the sake of brevity, we will not report responses of the number of single-product firms in the rest of the paper.

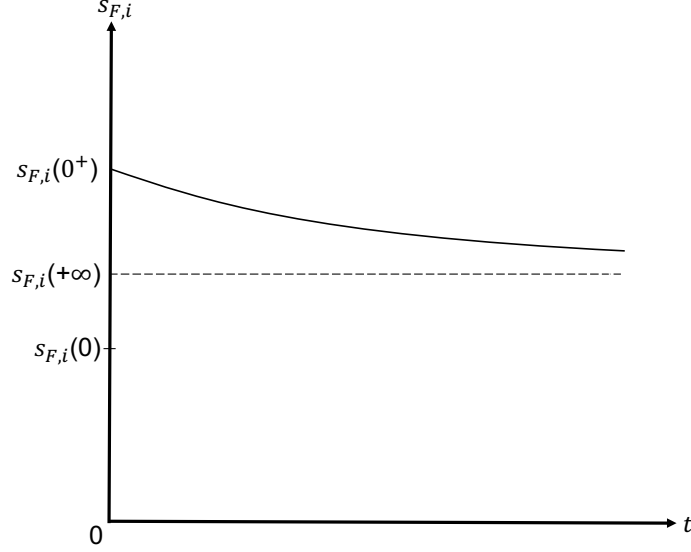


Figure 2: Dynamics of the foreign market share in response to a one-time increase in n_i

on impact and gradually declines afterwards, yet it remains higher than the initial share along the entire path.²⁶ The initial rise in the foreign market share is driven by an increase in the number of exported products, resulting from the larger product span, while the subsequent gradual decline is driven by a decrease in the product span of its foreign subsidiaries. In this case too, the foreign market share responds in a non-monotonic fashion: it rises first and declines subsequently.

This dynamic pattern is exhibited in Figure 2. The initial steady-state foreign market share is $s_{F,i}(0)$. It jumps on impact to $s_{F,i}(0^+)$ and gradually declines afterwards to $s_{F,i}(+\infty)$. Moreover, the one-time rise in n_i raises on impact the domestic market share that stays constant ever after.

Since markups are rising in market shares, it follows that firm i 's domestic markup rises on impact and remains at this higher level for the rest of time, while the foreign markups rise on impact and gradually declines over time. Nevertheless, the foreign markups remain higher than their initial values at every point in time after the initial adjustment. Exports relative to subsidiary sales rise on impact and keep rising after the initial shock, until the firm reaches its new steady state (see (11)).

These findings are summarized in

Proposition 5. *Suppose that n_i is constant and MNE i is in a steady state. Then a one-time increase in n_i : (i) raises on impact i 's domestic market share, price and markup, which remain constant afterwards; (ii) raises on impact i 's foreign market share, prices and markups, which gradually decline afterwards, but remain higher at all times than their initial values; (iii) gradually contracts the product span of its foreign affiliates; (iv) raises on impact i 's exports relative to subsidiary sales and gradually raises it afterwards.*

²⁶This is seen from the fact that $\zeta_{m,i}$ remains below its initial value on the entire trajectory, including the new steady state.

Evidently, a one-time rise in n_i sets in motion non-monotonic dynamics in the foreign market and a one-time adjustment in the home country. Importantly, it leads to a gradual decline in the range of products manufactured by foreign affiliates, suggesting that innovation and FDI are *substitutes*. The non-monotonicity of the foreign market share, depicted in Figure 2, interacts with this relationship. An increase in n_i raises on impact the MNE's foreign market share, which reduces the value of its marginal foreign product line. As a result, profitability of foreign investment declines. Combined with the natural attrition of all product lines, the lower profitability of expanding the foreign product scope leads to a gradual decline in the number of foreign product lines, and, as a consequence, to a gradual decline in the foreign market share.

We have so far pointed out non-monotonic responses over time to changes in the economic environment. Yet non-monotonicity can also arise in cross-sectional relationships. To this end we study correlations across large multinationals with different productivity levels.

Consider an improvement in MNE i 's labor productivity $1/a_i$ (a decline in a_i). This cost-saving raises the marginal profitability of foreign direct investment, given a foreign market share. Yet it also raises the foreign market share, which reduces marginal profits from FDI.²⁷ We show in the appendix that the former effect dominates if and only if $s_{F,i} < s_c$, where s_c is implicitly defined by

$$(\sigma - \delta s_c - 1)^2 [\sigma^2 - (\delta s_c)^2] = (\sigma - 1) (\delta s_c)^2. \quad (21)$$

When $s_{F,i} < s_c$, the $\dot{\zeta}_{m,i} = 0$ curve in Figure 1 shifts upward. In the opposite case, i.e., $s_{F,i} > s_c$, it shifts downward. In both cases, the $\dot{m}_i = 0$ curve does not change.

For $s_{F,i} < s_c$, the value of a foreign product line, $\zeta_{m,i}$, jumps up to a new equilibrium trajectory. Subsequently, $\zeta_{m,i}$ declines and m_i rises. In this case the foreign market share rises on impact (see appendix) and keeps rising afterwards, the foreign markup rises on impact and keeps rising after that, and exports relative to subsidiary sales gradually decline. These adaptations continue until the firm reaches a new steady state. As is evident, in these circumstances the dynamics are monotonic.

In the opposite case, i.e., when $s_{F,i} > s_c$, $\zeta_{m,i}$ jumps down to a new equilibrium trajectory. After that $\zeta_{m,i}$ gradually rises while m_i gradually declines. The foreign market share rises on impact and gradually declines afterwards. As a result, foreign markups rise on impact and gradually decline afterwards. Finally, during the transition to the new steady state, exports rise relative to subsidiary sales. In contrast to the case $s_{F,i} < s_c$, now there are non-monotonic responses.

Finally, independently of the size of the foreign market share, a productivity improvement raises on impact the firm's market share and markup in the home country, which remain constant thereafter.

An interesting implication of these results is that, if large firms differ in labor productivity but have a common product span, i.e., $n_i = n_j$ for all $i, j = 1, 2, \dots, I$, then in the steady state more

²⁷Marginal profits of FDI are $(1 - \tau^{1-\sigma}) \Lambda_i(s_{F,i})$. A fall in a_i shifts up $\Lambda_i(s_{F,i})$ for every $s_{F,i}$ and raises $s_{F,i}$. Since $\Lambda_i(s_{F,i})$ is a decreasing function, the rise in the market share reduces marginal profits of FDI. For this reason the effect on marginal profits on profitability of product lines of foreign affiliates can be positive or negative. See appendix for details.

productive firms have larger market shares at home and abroad. In spite of that, if the spread in labor productivity is wide enough, so that $s_{F,i} < s_c$ for some of the least productive firms and $s_{F,i} > s_c$ for some of the most productive firm, there will be a non-monotonic steady state relation between labor productivity and product scope of foreign subsidiaries, forming an inverted-U shape relationship.²⁸

Whenever there is an inverted-U-shaped relationship between labor productivity and product span of foreign subsidiaries, it leads to a U-shaped relationship between labor productivity and exports relative to subsidiary sales. In other words, controlling for a firm's total product span, n_i , exports relative to subsidiary sales first decline and later rise with labor productivity.²⁹ Naturally, if $s_{F,i} < s_c$ is satisfied for all large firms, then exports relative to subsidiary sales are negatively correlated with labor productivity in the cross section. Alternatively, if $s_{F,i} > s_c$ for all large firms, the correlation is positive. We summarize these findings in

Proposition 6. *Suppose that n_i is constant, the same for every $i = 1, 2, \dots, I$, and every MNE is in a steady state. Also suppose that dispersion of productivity across firms, $1/a_i$, is large enough, so that $s_{F,i} < s_c$ for some low-productivity firms and $s_{F,i} > s_c$ for some high-productivity firms, where s_c is characterized by (21). Then in the cross-section of large firms the relationship between productivity and product span of foreign affiliates has an inverted-U shape while the relationship between productivity and exports relative to subsidiary sales has a U shape.*

In this section we analyzed FDI dynamics, treating as exogenous the large firms' product spans. However, large firms can often engage in research and development in order to expand their product scope. When this is possible, a firm's labor productivity shapes its long-run overall product span as well as the product span of its foreign affiliates. Note from Proposition 5 that the larger is a firm's product span, the smaller is the long-run product span of its foreign affiliates (our substitutability result). This association interacts with the above-discussed correlations in forming the relationship between labor productivity and the product span of foreign affiliates, when n_i is endogenous. We examine these issues in Section 6. In the next section we illustrate the use of the model for an analysis of mergers.

5 Merger

To illustrate how the model can be used in various applications, we discuss in this section the impact of a merger between two large firms. For concreteness, and without loss of generality, suppose that

²⁸Feenstra and Ma (2008) were the first to derive an inverted-U shape relationship between firm productivity and product scope, in a static model of monopolistic competition with multi-product firms (see their Lemma 6.3). Raff and Wagner (2013) provided empirical support for this relationship with German firm-level data. Helpman and Niswonger (2021) showed that this type of relationship also arises in a dynamic closed economy, populated by large multi-product firms and a fringe of small single-product firms. They also provided empirical support for this relationship with U.S. firm-level data. Finally, Macedoni (2022) documented an inverted-U shape relation between market shares and product scope in export data from 11 countries.

²⁹We will see in the next section that this relation is modified when n_i is endogenous.

firms 1 and 2 merge. This merger does not change the business strategy of the other large firms, because the price indexes P_H and P_F do not change due to continued entry of single-product firms.

For simplicity, assume that the combined firm has to produce every product in spans n_i and m_i with the technology a_i , and we focus on the case in which n_1 and n_2 are exogenous, as in the previous section. We first consider an instantaneous equilibrium. In this case the merged firm chooses prices to maximize joint profits from product spans n_i and m_i , $i = 1, 2$, for given values of these product spans. When the merged firm makes these decisions, it also takes as given the number of single-product firms that have entered the industry, their prices and prices of all the other large multi-product firms. Accordingly, the merged firm solves the following problem:

$$\max_{\{p_i, p_{e,i}, p_{m,i}\}_{i=1,2}} \sum_{i=1}^2 n_i P_H^\delta p_i^{-\sigma} (p_i - a_i) + \sum_{i=1}^2 (n_i - m_i) P_F^\delta p_{e,i}^{-\sigma} (p_{e,i} - \tau a_i) + \sum_{i=1}^2 m_i P_F^\delta p_{m,i}^{-\sigma} (p_{m,i} - a_i)$$

subject to (4) and (5). We show in the appendix that this leads the merged firm to price products in the home country according to

$$p_i = \frac{\sigma - \delta s_{1+2}}{\sigma - \delta s_{1+2} - 1} a_i, \quad i = 1, 2, \quad (22)$$

where s_{1+2} is the market share of the firm from sales of all its products. This means that the firm charges in the home country the same markup on all its products, and that this markup is an increasing function of the combined market share s_{1+2} ; i.e., $p_i(s_{1+2})$ is an increasing function. Moreover, since

$$s_{1+2} = \frac{\sum_{i=1}^2 n_i p_i^{1-\sigma}}{P_H^{1-\sigma}},$$

the last two equations provide a solution to the market share s_{1+2} as a function of n_1 and n_2 , and therefore the home price of a product in n_i is also a function of n_1 and n_2 . Since these total product spans are fixed, the merged firm adjusts domestic prices once and keeps them constant after that. As is quite intuitive, the merger raises the firm's combined market share above the market shares of each one of the stand-alone firms, as a result of which its markups are higher and so are its prices. In short, a merger leads to escalation of markups and prices in the home country.

Next, consider the foreign market, where $s_{F,1+2}$ is the combined market share from all products of the merged firm's exports and subsidiary sales. We show in the appendix that the firm's export prices, $p_{e,i}$, and prices of subsidiary sales, $p_{m,i}$, satisfy

$$p_{e,i} = \frac{\sigma - \delta s_{F,1+2}}{\sigma - \delta s_{F,1+2} - 1} \tau a_i, \quad i = 1, 2,$$

$$p_{m,i} = \frac{\sigma - \delta s_{F,1+2}}{\sigma - \delta s_{F,1+2} - 1} a_i, \quad i = 1, 2,$$

where the market share is given by

$$s_{F,1+2} = \frac{\sum_{i=1}^2 (n_i - m_i) p_{e,i}^{1-\sigma} + \sum_{i=1}^2 m_i p_{m,i}^{1-\sigma}}{P_F^{1-\sigma}}.$$

Note that these pricing equations imply that the export price of a product in n_i is τ times higher than the price charged by a foreign affiliate for products in m_i . For that reason markups are the same within this group of products, independently of whether they are exported or supplied by foreign subsidiaries. Moreover, and quite intuitively, we show in the appendix that the combined markup s_{1+2} is larger than the markup of either one of the stand-alone firms 1 or 2. That being the case, the merged firm charges higher markups and higher prices in the foreign market than either one of the stand-alone firms would have done.

The last three equations provide a solution to $\{p_{i,e}, p_{i,m}\}$ for $i = 1, 2$ and $s_{F,1+2}$. The combined foreign market share is a function of n_1, n_2, m_1 and m_2 , which we express as $s_{F,1+2}(n_1, n_2, m_1, m_2)$. Since prices charged by foreign affiliates for products in m_i are a function of $s_{F,1+2}(n_1, n_2, m_1, m_2)$, we can express these prices as

$$p_{i,m}(n_1, n_2, m_1, m_2) := p_i[s_{F,1+2}(n_1, n_2, m_1, m_2)] \text{ for } i = 1, 2.$$

The merged firm adjusts foreign prices instantly, both for exports and for subsidiary sales, and foreign prices change over time in response to foreign direct investment.

We can use these characterizations of prices and market shares to formulate the merged firm's dynamic decision concerning foreign direct investment. The firm maximizes the discounted present value of operating profits minus FDI costs. We show in the appendix that, using $\zeta_{m,1}$ and $\zeta_{m,2}$ to denote the co-state variables of equations that describe the dynamics of foreign product spans, (15), the steady state conditions for the merged firm can be described by

$$(r + \theta) \zeta_{m,i} = (1 - \tau^{1-\sigma}) \Lambda_i[s_{F,1+2}(n_1, m_1, n_2, m_2)], \quad i = 1, 2, \quad (23)$$

$$\phi_m \left[\varphi_m \left(\zeta_{m,i}^{-1} \right) \right] = \theta m_i, \quad i = 1, 2, \quad (24)$$

where

$$\frac{\partial \pi_{F,1+2}}{\partial m_i} = (1 - \tau^{1-\sigma}) \Lambda_i[s_{F,1+2}(n_1, m_1, n_2, m_2)], \quad i = 1, 2,$$

and, as before,

$$\Lambda_i(s) := \left(\frac{\sigma - \delta s}{\sigma - \delta s - 1} \right)^{-\sigma} \frac{a_i^{1-\sigma} P^\delta \sigma}{(\sigma - \delta s - 1) \sigma + (s\delta)^2}, \quad \Lambda'_i(s) < 0$$

and $\pi_{F,1+2}$ represents profits net of FDI costs of the merged firm. Since the merged firm has a larger foreign market share than either one of the stand-alone firms 1 or 2, (23) implies that the value of a marginal product line of m_i , $\zeta_{m,i}$, is lower in the steady state of the merged firm than either of the stand-alone firms 1 or 2 would have done. That being the case, (24) implies that the merged firm

engages in less FDI and has a lower product span m_i in the long run. With lower scopes of foreign products, exports relative to subsidiary sales rise.

Intuitively, a merger raises the combined firm's market share above the level of each one of the individual firms'. This reduces the value of a marginal product line in foreign subsidiaries, thereby reducing the incentives to engage in foreign direct investment. Lower FDI leads in turn to fewer product lines in foreign subsidiaries. This illustrates yet another interesting manifestation of substitutability between the total product span and the product spans of foreign affiliates.

Can we assess the welfare implications of a merger? With quasi-linear preferences, the indirect utility function is (see footnote 9)

$$v = l + y + \frac{1}{\varepsilon - 1} P^{1-\varepsilon},$$

where l is labor income, y is other types of income, and the last term on the right-hand side represents consumer surplus. Assuming no government intervention, y equals aggregate profits net of FDI costs of the large firms $i = 1, 2, \dots, I$.

Consumers are interested in the discounted utility flow. Assume that the subjective discount rate equals the interest rate r , this discounted utility flow is

$$V = \int_0^\infty e^{-rt} v(t) dt,$$

where $v(t)$ is the utility flow at time t . Note, however, that labor income and consumer surplus do not vary over time. For this reason the home country benefits from a merger if and only if the merger raises the present value of aggregate operating profits minus FDI costs. Since a merger between firms 1 and 2 has no impact on the business strategies of other large firms, the merger is beneficial if and only if the present value of its operating profits minus FDI costs exceed the combined present value of the operating profits minus FDI costs of the stand-alone firms 1 and 2. The merged firm can choose the same business strategies that would have been chosen by the individual firms 1 and 2, yet it prefers to charge higher markups and higher prices, and eventually have foreign affiliates with fewer product lines. Therefore, by revealed preference, the merged firm generates a higher present value of operating profits net of FDI costs than the two stand-alone firms together would have done. In short, the merger is beneficial.

Entry of single-product firms at every instant plays a vital role in this result. While a merger leads to higher markups and higher prices in the domestic market, which harm consumers, this is compensated for by entry of single-product firms that prevents these higher prices to raise the price index P_H . On the other side, the higher prices at home and abroad, and the modified FDI strategy, raise the discounted present value of income. The end result is more consumption of the homogeneous good without reducing real consumption of varieties of the differentiated product. Ergo the welfare gain.

6 Innovation and FDI

In this section we endow large firms with an innovation technology, without changing other details of the model. A firm can use this technology to augment its product span, similar to Klette and Kortum (2004). As a result, firm i can invest in either expanding n_i or m_i . An investment flow of $\iota_{n,i}$ per unit time expands n_i by $\phi_n(\iota_{n,i})$ units per unit time. The innovation function $\phi_n(\iota_{n,i})$ is increasing, concave, $\phi_n(0) = 0$ and it satisfies the Inada conditions $\lim_{\iota \rightarrow 0} \phi'_n(\iota) = +\infty$ and $\lim_{\iota \rightarrow +\infty} \phi'_n(\iota) = 0$. Furthermore, n_i depreciates at the rate θ per unit time, randomly hitting every available brand. For these reasons n_i satisfies the differential equation

$$\dot{n}_i = \phi_n(\iota_{n,i}) - \theta n_i, \text{ for all } t \geq 0. \quad (25)$$

When a brand is hit by a negative shock, the good cannot be exported anymore, and if it was also produced by a foreign affiliate, subsidiary sales of this product come to a halt.

In this environment, firm i solves an optimal control problem in which m_i and n_i are state variables, while FDI, $\iota_{m,i}$, and investment in innovation, $\iota_{n,i}$, are control variables, facing the initial conditions $m_i(0) = m_i^0$ and $n_i(0) = n_i^0$. This problem is spelled out in the appendix. Compared to the optimal control problem from the previous section, it has an additional state variable, n_i , and an additional co-state variable, $\zeta_{n,i}$, affiliated with the differential equation for n_i . The first-order conditions of this problem yield four differential equations,³⁰

$$\dot{\zeta}_{m,i} = (r + \theta) \zeta_{m,i} - (1 - \tau^{1-\sigma}) \Lambda_i \{s_i [(n_i - m_i) \tau^{1-\sigma} + m_i]\}, \quad (26)$$

$$\dot{m}_i = \phi_m \left[\varphi_m \left(\zeta_{m,i}^{-1} \right) \right] - \theta m_i, \quad (27)$$

$$\dot{\zeta}_{n,i} = (r + \theta) \zeta_{n,i} - \Lambda_i [s_i(n_i)] - \tau^{1-\sigma} \Lambda_i \{s_i [(n_i - m_i) \tau^{1-\sigma} + m_i]\}, \quad (28)$$

$$\dot{n}_i = \phi_n \left[\varphi_n \left(\zeta_{n,i}^{-1} \right) \right] - \theta n_i, \quad (29)$$

and two transversality conditions

$$\lim_{t \rightarrow \infty} e^{-rt} \zeta_{m,i}(t) m_i(t) = 0,$$

$$\lim_{t \rightarrow \infty} e^{-rt} \zeta_{n,i}(t) n_i(t) = 0.$$

The first two differential equations are the same as (17) and (18); they describe the dynamics of the product scope of foreign affiliates, m_i , and its co-state variable, $\zeta_{m,i}$, which represents the value of a marginal product line in foreign subsidiaries. Recalling the detailed discussion of the interpretation of these conditions in the previous section, note that (26) represents a no-arbitrage condition for asset pricing, where the asset is a foreign subsidiary product line. The condition says that profits from this product line plus expected capital gains—consisting of the appreciation of

³⁰ As in the previous section, we assume an interior solution. That is, $n_i > 0$ and $m_i \in (0, n_i)$ at every point in time.

the value of the asset minus the expected total loss from the possibility that the product line will become obsolete—equal the interest income that the asset owner can earn for sure from an interest bearing deposit equal to the value of the asset. Condition (27) states that, given the value of a marginal foreign product line, the expansion of the foreign product scope equals the addition of new product lines that results from optimal foreign direct investment, minus the reduction of available product lines due to depreciation.

Differential equations (28)-(29) have similar interpretations for the dynamics of the total product span, n_i , and its co-state variable, $\zeta_{n,i}$, that stands for the value of a marginal product line available to the firm. Equation (28) represents the no-arbitrage condition for the pricing of a marginal product line in n_i . Here the profit stream consists of the contribution of the product line to profits from domestic sales, $\Lambda_i[s_i(n_i)]$, plus the contribution to profits from exports, $\tau^{1-\sigma} \Lambda_i\{s_i[(n_i - m_i)\tau^{1-\sigma} + m_i]\}$. The expected capital gain consists of the appreciation value $\dot{\zeta}_{n,i}$ minus the expected total capital loss $\theta\zeta_{n,i}$. The interest earnings on a deposit equal to the value of the asset are $r\zeta_{n,i}$. Combining these elements, the equation states that the income flow plus the expected capital gains generated by the asset equal potential interest earning, which makes an investor indifferent between holding the asset and using its value to earn interest income.

Equation (29) describes the dynamics of the total product span n_i . The value of a marginal product line determines investment in innovation, $\iota_{n,i} = \varphi_n(\zeta_{n,i}^{-1})$, which results from the optimal innovation policy that equates the last dollar of investment with the marginal value of the product lines generated by this investment.³¹ This investment level determines in turn the gross addition to the total product scope, $\phi_n[\varphi_n(\zeta_{n,i}^{-1})]$. The net increase in the number of available products equals the gross addition minus the attrition, θn_i . For this reason (29) describes the evolution of the total product scope of the firm.

The first transversality condition is the same as the transversality condition in the previous section while the second transversality condition applies to the dynamics of n_i . A trajectory that leads to a steady state satisfies both conditions.

We show in the appendix that this dynamic system satisfies local saddle-path stability around the steady state. That is, we show that the 4×4 matrix of the linearized system of the differential equations, evaluated at the steady state values of m_i , n_i , $\zeta_{m,i}$ and $\zeta_{n,i}$, has two positive and two negative characteristic roots. As a result, the four free coefficients of the general solution to this linear system of differential equations can be chosen to yield $m_i(0) = m_i^0$ and $n_i(0) = n_i^0$, on one hand, and initial values of $\zeta_{m,i}(0)$ and $\zeta_{n,i}(0)$ that place the system on a path that converges to the steady state, on the other.

In a steady state $\dot{\zeta}_{m,i} = \dot{\zeta}_{n,i} = \dot{m}_i = \dot{n}_i = 0$. From $\dot{m}_i = 0$ we can solve $\zeta_{m,i}$ as a function of m_i , $\zeta_{m,i} = \psi_{m,i}(m_i)$, while from $\dot{n}_i = 0$ we can solve $\zeta_{n,i}$ as a function of n_i , $\zeta_{n,i} = \psi_{n,i}(n_i)$, where both functions are increasing in their arguments. Substituting these functions into $\dot{\zeta}_{m,i} = \dot{\zeta}_{n,i} = 0$ yields

$$(r + \theta) \psi_{m,i}(m_i) - (1 - \tau^{1-\sigma}) \Lambda_i\{s_i[(n_i - m_i)\tau^{1-\sigma} + m_i]\} = 0, \quad (30)$$

³¹As shown in the appendix, the optimal choice of $\iota_{n,i}$ yields the first-order condition $\zeta_{n,i}\phi'(\iota_{n,i}) = 1$, from which we obtains $\iota_{n,i} = \varphi_n(\zeta_{n,i}^{-1})$.

$$(r + \theta) \psi_{n,i}(n_i) - \Lambda_i[s_i(n_i)] - \tau^{1-\sigma} \Lambda_i\{s_i[(n_i - m_i)\tau^{1-\sigma} + m_i]\} = 0. \quad (31)$$

The first equation describes a relationship between m_i and n_i that secures FDI at a level that keeps m_i constant and a discounted present value of marginal profits from FDI that motivates the firm to engage in this investment level.³² Intuitively, it represents asset pricing in a stationary environment, in which the profit flow is constant and the value of the asset does not change, the asset being a marginal product line in the foreign affiliates. This relationship is depicted by the downward sloping *MM* curve in Figure 3, in line with the result in Proposition 5(iii). Sliding down the *MM* curve raises m_i and reduces n_i , but also reduces $n_{F,i}$ —the effective number of products sold in F —and the foreign market share. While the foreign market share rises in response to the expansion of the product span of foreign subsidiaries, the decline in the range of exported products has a larger negative effect on $n_{F,i}$, and therefore on the foreign market share. The decline of n_i reduces the domestic market share and therefore the foreign and domestic market shares are positively correlated along the *MM* curve.

The second equation describes a relationship between m_i and n_i which secures an R&D level that keeps n_i constant and a discounted present value of marginal profits from R&D that motivates the firm to engage in this level of investment. It too represents asset pricing in a stationary environment, in which the profit flow is constant and the value of the asset does not change, except that now the asset is a marginal product line in the total product span. This relationship is depicted by the downward sloping *NN* curve in Figure 3. The *MM* curve is steeper at the intersection point of the two curves, as depicted in the figure (see appendix for a proof). Sliding up the *NN* curve raises n_i and reduces m_i , but also reduces $n_{F,i}$ and the foreign market share. While the foreign market share rises in response to an increase in the number of exported goods, it declines in response to the contraction of the number of products manufactured by foreign affiliates and the latter effect dominates. Since an increase in n_i raises the domestic market share, it follows that the domestic and foreign market shares are negatively correlated along the *NN* curve. The intersection point between the *MM* and *NN* curves at point e describes a long-run equilibrium.

In the previous section we analyzed the impact of a productivity improvement in FDI, treating as exogenous the product spans n_i , $i = 1, 2, \dots, I$. We found that it raised the product span of foreign subsidiaries, their prices and markups, their foreign market shares, and reduced every large firm's exports relative to subsidiary sales (see Proposition 4(iii)). To analyze the impact of this type of productivity improvement in the current setting, where the product span is endogenous, we turn to Figure 3. A productivity improvement in FDI shifts the *MM* curve rightward and does not modify the *NN* curve.³³ This leads to a larger product span of foreign affiliates, but a smaller overall product span. That is, n_i declines in the long run. This is another manifestation of the inherent substitutability between m_i and n_i . While in the previous section—when n_i was constant—an improvement in the foreign investment technology had no impact on the domestic

³²The discounting includes the interest rate, r , and the attrition rate, θ . The latter represents a risk premium.

³³As in the previous section, we replace the function $\phi_m(\iota)$ with $z_m \phi_m(\iota)$, where $z_m = 1$ initially, and examine an increase in z_m .

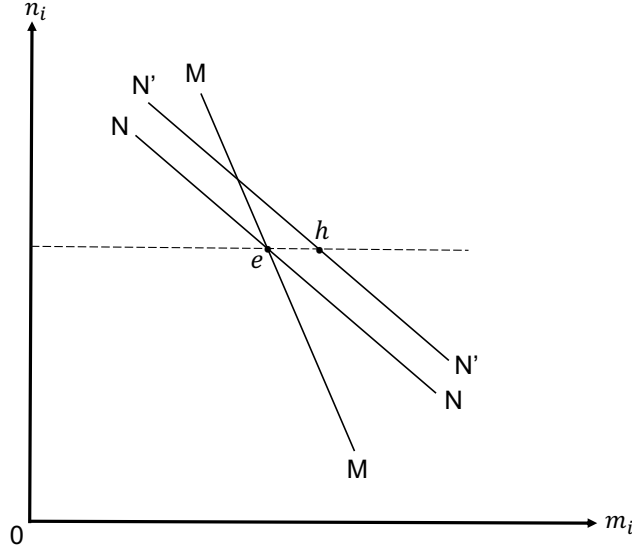


Figure 3: Steady state values of m_i and n_i

market share, now this technical change induces firm i to reduce its market share in the home country, and with it to reduce markups and prices. In contrast, the market share rises in the foreign country, because sliding down the NN curve reduces the domestic market share and raises the foreign market share. As a result, firm i 's foreign subsidiaries charge higher markups and prices. Furthermore, fewer products are exported and more products are produced by foreign affiliates. This leads to lower exports relative to subsidiary sales, in line with the proximity-concentration tradeoff theory of foreign direct investment.

Technical change that raises R&D efficiency shifts upward the NN curve, which leads to a larger product span, n_i , and fewer varieties manufactured by foreign subsidiaries, m_i . The firm expands its market share in the home country and raises markups and prices in that market. Along the MM curve the foreign and domestic market shares are positively correlated. Therefore this technical improvement raises the foreign market share, as well as foreign markups and prices. Moreover, since fewer products are produced by foreign subsidiaries and more products are exported, every large firm's exports rise relative to subsidiary sales.

We summarize these findings in

Proposition 7. *Suppose MNE i is in a steady state. Then: (i) an improvement in the FDI technology leads to higher m_i and lower n_i in the long run, a higher market share in the foreign country and a lower market share at home, higher markups and prices in the foreign country and lower markups and prices at home, and smaller exports relative to subsidiary sales; (ii) an improvement in the innovation technology leads to lower m_i and higher n_i in the long run, higher market shares in both countries, higher markups and prices in both countries, and larger exports relative to subsidiary sales.*

Interestingly, while technical improvements in the R&D technology generate positive correlations

between a firm's responses at home and abroad, and raise exports relative to subsidiary sales, technical improvements in the FDI technology generate negative correlations between the firm's responses at home and abroad, and reduce exports relative to subsidiary sales. Intuitively, these differences are driven by the substitutability between the product span n_i and the product scope of foreign affiliates. A better R&D technology leads naturally to a long-run increase in the total product span, which brings about a contraction of the product scope of foreign affiliates through the substitutability mechanism that we have repeatedly emphasized. These changes trigger the other adjustments described in part (ii) of Proposition 7. Similarly, a better FDI technology leads naturally to a long-run increase in the product span of foreign affiliates, which brings about a contraction of the total product scope of the firm through this same substitutability mechanism. And these changes trigger the other adjustments described in part (i) of Proposition 7.³⁴

Next consider labor productivity. Recall from the previous section that an improvement in firm i 's labor productivity, $1/a_i$, raises the marginal value of investment in the product span of foreign subsidiaries, $(1 - \tau^{1-\sigma}) \Lambda_i(s_{F,i})$, if and only if the foreign market share is small enough, i.e., $s_{F,i} < s_c$. As explained in the previous section, a smaller market share mutes the negative indirect effect of the productivity improvement (through its elevation of the foreign market share) on the value of a marginal product line in foreign subsidiaries compared to the positive direct effect. The same argument establishes that an improvement in firm i 's labor productivity raises the marginal value of investment in product span n_i , $\Lambda_i(s_i)$, if and only if the home country market share is small enough, i.e., $s_i < s_c$ (see appendix). When $\Lambda_i(s_{F,i})$ increases in response to an improvement in the firm's labor productivity, the MM curve in Figure 3 shifts to the right while the NN curve shifts upward. In contrast, when $\Lambda_i(s_i)$ increases in response to an improvement in the firm's labor productivity, the MM curve does not change while the NN curve shifts upward. It follows that when an improvement in labor productivity shifts upward both $\Lambda_i(s_{F,i})$ and $\Lambda_i(s_i)$, the MM curve shifts to the right while the NN curve shifts upward.

Figure 3 shows the initial equilibrium point e , where the NN and MM curves intersect, and the upward shift in the NN curve to $N'N'$ as a result of upward shifts in $\Lambda_i(s_{F,i})$ and $\Lambda_i(s_i)$ in response to a rise in labor productivity. Since $s_i > s_{F,i}$, condition $s_i < s_c$ is sufficient for $\Lambda_i(s_{F,i})$ and $\Lambda_i(s_i)$ to increase in response to a decline in a_i . We now argue that the rightward shift of the MM curve is to the left of point h . To see why, note from (31) that at point h the marginal value $\Lambda_i(s_{F,i})$ has to be lower than it was originally at e (i.e., before the change in labor productivity), because $\Lambda_i(s_i)$ is higher. For this reason, the left-hand side of (30) is positive at h and the rightward shift of MM stops short of point h . Under these circumstances the new equilibrium point is above the horizontal line through e and h , and for that reason product span n_i is larger in the new equilibrium (see appendix for an analytical proof). It is also clear from the figure that the product span of foreign subsidiaries may rise or decline in response to firm i 's improvement in labor productivity. A similar analysis shows that whenever $s_{F,i} > s_c$ in the initial equilibrium, and therefore $s_i > s_c$, an improvement in

³⁴We provide a discussion of the impact of higher export costs, τ , on long-run outcomes in the appendix. In particular, we provide conditions for an increase in these costs to raise the product scope of foreign affiliates and a reduction in the overall product span of a firm.

labor productivity induces firm i to contract its product span in the long run, but the firm may expand or contract the product span of its foreign subsidiaries.

In spite of the ambiguity concerning the response of m_i to changes in labor productivity, we show in the appendix that there is no ambiguity concerning the response of the effective number of products sold in the foreign country, $n_{F,i}$; it rises with labor productivity when $s_i < s_c$ and declines with labor productivity when $s_{F,i} > s_c$. We summarize these findings in

Proposition 8. *Suppose MNE i is in a steady state. Then an improvement in its labor productivity $1/a_i$: (i) raises n_i and $n_{F,i}$ in the long run when $s_i < s_c$; and (ii) reduces n_i and $n_{F,i}$ in the long run when $s_{F,i} > s_c$.*

These results imply that in the cross section of firms the relationship between labor productivity and long-run product spans n_i has an inverted-U shape whenever the smallest value of labor productivity secures a domestic market share smaller than s_c and the largest value secures a foreign market share larger than s_c .³⁵ Moreover, the proposition implies that in these circumstances an inverted-U relationship also exists between the effective number of products supplied to the foreign country, $n_{F,i}$, and labor productivity.

Improvements in firm i 's labor productivity directly raise its market shares in both countries when the (effective) numbers of products sold in those countries do not change. Additionally, the firm's market share in a particular country is larger the larger is the (effective) number of products sold there. Therefore, as long as $s_i < s_c$, higher labor productivity raises the firm's market share at home and abroad, because both the direct effect of labor productivity and its indirect effect (through the number of products) raise these market shares. In contrast, when $s_{F,i} > s_c$, the direct effect raises market shares at home and abroad while the indirect effect reduces them. We nevertheless show in the appendix that in this case the market share rises at home or abroad, and we find in our simulations that both market shares tend to rise with productivity.³⁶

We illustrate in Figure 4 a set of steady-state relationships between market shares, product spans and exports relative to subsidiary sales.³⁷ In this example market shares rise with labor productivity. At the same time, both the overall product span and the product span of foreign affiliates have an inverted-U shape. Furthermore, in this example, exports relative to subsidiary sales have an inverted-U shape (with variation in a band smaller than five percentage points). The latter contrasts with the finding in the previous section, where n_i was exogenous and constant, of a U-type relation between productivity and exports relative to subsidiary sales. The difference is caused by the endogeneity of n_i . When both the overall product span and the product span of foreign affiliates change in the same direction, the ratio n_i/m_i —which drives exports relative to subsidiary sales—may rise or decline. In the example, it produces the inverted-U shape. This finding illustrates the importance of endogenous innovation in shaping the long-run relationship between labor productivity of firms and their exports relative to subsidiary sales.

³⁵Clearly, if $s_c > 1$ there is no inverted-U relationship, and more productive firms have larger product spans in the long run.

³⁶In Helpman and Niswonger (2021) we show that in a closed economy the market share always rises.

³⁷In this example $\sigma = 4$, $\varepsilon = 1.5$, $r = 0.05$, $\theta = 0.05$, $\tau = 1.15$, $P = 0.8$, $\phi_m(\iota) = 0.75\sqrt{\iota}$, $\phi_n(\iota) = 0.1\sqrt{\iota}$.

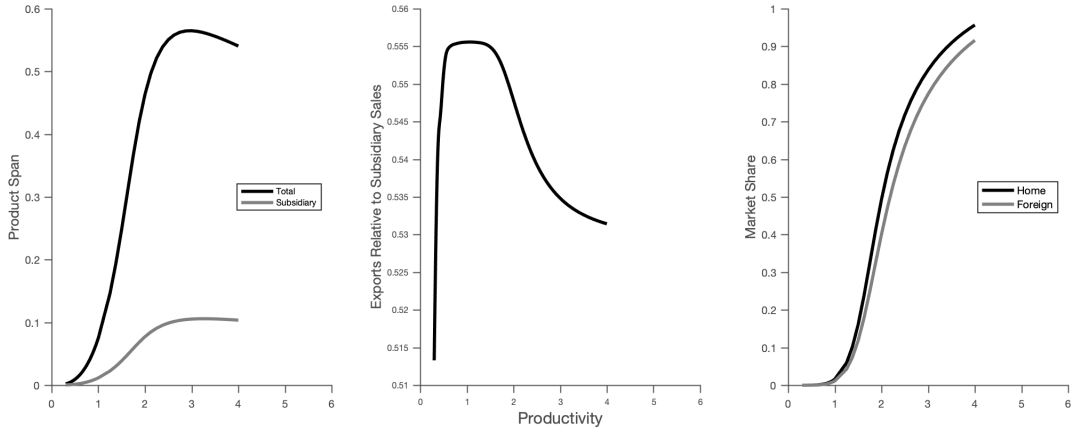


Figure 4: Steady-state relationships

7 Summary and Conclusions

Multinational enterprises play an outsize role in the world’s economy. They are large employers of labor at home and abroad. Their value added is enormous, they dominate international trade, and they spend large sums of money on research and development. Furthermore, they are multi-product firms that serve foreign markets with both exports and sales of foreign affiliates.

We have developed in this paper a model of large MNEs that captures these salient features. Because of the size of these enterprises, we view them as oligopolists that compete at home and abroad. But consistent with the evidence described in the Introduction, they also face competition from small single-product firms who form a monopolistically competitive fringe. Using this framework, we analyze the dynamics of foreign direct investment and product innovation. To show the usefulness of the model, we apply it to a merger, and explore the dynamics of the merged firm and its welfare consequences.

Our analysis highlights two themes. First, expansion of the scope of a firm’s products is a substitute for the expansion of the product scope of its foreign affiliates. In other words, more overall product scope leads to less product scope of foreign subsidiaries. Second, there are two types of non-monotonicity: in the time series and in the cross-section. A change in the economic environment can generate short-run responses that differ from long-run trends. And in the cross-section of firms that differ in productivity, some economic statistics may be non-monotonically related to productivity levels.

To aid the understanding of our findings, we first examined in Section 4 the simpler case of firms with fixed overall product scopes (no product innovation). We showed that an exogenous increase in the firm’s product scope reduces profitability of a marginal product line in foreign affiliates, triggering a decline in FDI and a gradual decline in the product span of foreign affiliates. This demonstrates the substitutability between the overall product scope and the product scope of foreign affiliates. We also showed that the firm’s foreign market share rises in response to an

increase in its product scope (due to an expansion of exports at the extensive margin), but that the foreign market share contracts subsequently in response to the decline in the product span of foreign affiliates. This demonstrates a non-monotonic response. In similar vein, the firm’s markups—of exports and subsidiary sales—rise initially and decline subsequently, and prices follow a similar pattern. But not all responses are non-monotonic. The larger product scope leads to an immediate expansion of exports relative to subsidiary sales, and this statistic keeps rising afterwards. We also showed in that section that a large dispersion of productivity levels across firms leads to an inverted-U shape relationship between productivity and the product span of foreign affiliates. This demonstrates a non-monotonic cross-sectional relationship.

We also discussed in Section 4 the impact of export costs and FDI costs. Of particular interest is the observation that changes in export costs generate non-monotonic responses, while changes in FDI costs set in motion monotonic adjustments over time. To illustrate, an increase in export costs reduces the foreign market share in the short-run, which rises subsequently as exports are replaced by a growing product span of foreign affiliates. In contrast, a decline in FDI costs does not bring about a change in the foreign market share on impact, but leads to its gradual increase over time. Brexit represents an episode in which trade and FDI costs have increased for British companies, while Mexico’s 1985 liberalization of foreign direct investment represents an episode in which FDI costs declined for American firms.

Our model can be used for various applications, which we illustrated with an analysis of a merger between two large firms (see Section 5).³⁸ We showed that in the long run the merged firm has a larger market share than either one of its component firms, and that it values less a marginal product line in its foreign affiliates. For this reason the merged firm invests less in foreign product lines and maintains lower product scopes of the foreign subsidiaries. The substitutability between a firm’s product scope and the product span of its foreign affiliates plays a key role in this outcome. The merged firm raises markups and prices at home and abroad, as well as profits. Despite the hike in domestic prices, home welfare improves, because entry of single-product firms offsets their impact on the price index of real consumption.

Substitutability between a firm’s product scope and the product span of its foreign affiliates plays a key role in the joint determination of FDI and innovation, discussed in Section 6. In that section we showed that a reduction in FDI costs raises the product span of foreign affiliates in the long run and reduces a firm’s overall product span. As a result, a MNE raises foreign markups and prices, but reduces domestic markups and price. Its foreign market share rises while its domestic market share declines. Furthermore, exports relative to subsidiary sales decline. In contrast, an improvement in the R&D technology reduces the foreign product span in the long run and increases the MNE’s overall product scope. This leads to higher market shares, markups and prices at home and abroad, and to larger exports relative to subsidiary sales. We also showed that with the joint determination of FDI and innovation, the cross-sectional relationship between firm-level product spans and labor productivity can have an inverted-U shape. Moreover, the relationship between

³⁸The analysis assumed constant overall product spans of the merged firm’s component firms.

product spans of foreign subsidiaries and labor productivity and the relationship between exports relative to subsidiary sales and labor productivity can have an inverted-U shape.

Our analysis has generated a set of findings that can be tested with suitable data. There is, however, a paucity of data sets with detailed information about products that MNEs manufacture at home and abroad, which are needed for this type of empirical analysis. Yet there has been progress on this front, and Head and Mayer (2019) provide a good example of this progress for the car industry. When better data sets become available, it will be possible to test predictions of our model concerning the evolution of product spans of large multinationals and the product spans of their subsidiaries. In the meantime it may be possible to examine implications of the theory that do not require such detailed data. This may include responses of exports relative to subsidiary sales to changes in trade costs, FDI costs or productivity improvements. Our analytical framework can also be extended to include multiple sectors and multiple countries as well as simple forms of comparative advantage. Such an extended framework will provide predictions for variations across sectors and countries, which may be more amenable to empirical analysis. More challenging, however, would be to include platform FDI and time-varying stochastic shocks. All these remain topics for future research.

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APPENDIX

A Instantaneous Equilibrium

As described in the main text, large multi-product firms recognize the impact of their pricing policy on the price indexes P_F and P_H . For this reason firm i , which seeks to maximize profits, solves the following problem:

$$\max_{p_i, p_{e,i}, p_{m,i}} n_i P_H^\delta p_i^{-\sigma} (p_i - a_i) + (n_i - m_i) P_F^\delta p_{e,i}^{-\sigma} (p_{e,i} - \tau a_i) + m_i P_F^\delta p_{m,i}^{-\sigma} (p_{m,i} - a_i)$$

subject to (4)-(5). The first-order conditions can be expressed as

$$\delta \frac{n_i p_i^{1-\sigma}}{P_H^{1-\sigma}} = \sigma - \frac{p_i}{p_i - a_i},$$

$$\delta \frac{(n_i - m_i) p_{e,i}^{1-\sigma}}{P_F^{1-\sigma}} = \frac{(n_i - m_i) p_{e,i}^{-\sigma} (p_{e,i} - \tau a_i)}{(n_i - m_i) p_{e,i}^{-\sigma} (p_{e,i} - \tau a_i) + m_i p_{m,i}^{-\sigma} (p_{m,i} - a_i)} \left(\sigma - \frac{p_{e,i}}{p_{e,i} - \tau a_i} \right),$$

$$\delta \frac{m_i p_{m,i}^{1-\sigma}}{P_F^{1-\sigma}} = \frac{m_i p_{m,i}^{-\sigma} (p_{m,i} - a_i)}{(n_i - m_i) p_{e,i}^{-\sigma} (p_{e,i} - \tau a_i) + m_i p_{m,i}^{-\sigma} (p_{m,i} - a_i)} \left(\sigma - \frac{p_{m,i}}{p_{m,i} - a_i} \right).$$

Since $n_i p_i^{1-\sigma} / P_H^{1-\sigma}$ equals the share of firm i in country H , s_i , we have (9). The first of the above first-order conditions then yields (6). The last two first-order conditions imply

$$\frac{p_{e,i}}{p_{e,i} - \tau a_i} = \frac{p_{m,i}}{p_{m,i} - a_i},$$

and therefore $p_{e,i} = \tau p_{m,i}$. The firm's foreign market share from exports is $(n_i - m_i) p_{e,i}^{1-\sigma} / P_F^{1-\sigma}$ and its foreign market share from subsidiary sales is $m_i p_{m,i}^{1-\sigma} / P_F^{1-\sigma}$. Adding up these shares yields the combined foreign market share, $s_{F,i}$, described in (10). Next, adding up the last two first-order equations from above, using $p_{e,i} = \tau p_{m,i}$, yields

$$\delta s_{F,i} = \sigma - \frac{p_{m,i}}{p_{m,i} - a_i},$$

which yields (8). The pricing equation for exporters, (7), is then obtained from $p_{e,i} = \tau p_{m,i}$.

B Properties of Shares and Prices

Recall that $p_{e,i} = \tau p_{m,i}$. This allows us to rewrite equation (10) as

$$s_{F,i} = \frac{[(n_i - m_i) \tau^{1-\sigma} + m_i] p_{m,i}^{1-\sigma}}{P_F^{1-\sigma}}.$$

Noting that $p_{m,i}$ is solely a function of the share as seen in equation (8), we can see that $s_{F,i}$ and $p_{m,i}$ share the same relationship as s_i and p_i . The only difference is that the effective number of products is given by $n_{F,i} := (n_i - m_i) \tau^{1-\sigma} + m_i$ rather than by n_i . This allows us to write the foreign share as

$$s_{F,i} = s_i(n_{F,i}) = s_i[(n_i - m_i) \tau^{1-\sigma} + m_i],$$

where the function $s_i(n)$ is given by the joint solution to equations (9) and (6). Furthermore, this implies that because the price is simply a function of the share we have the analogous relationship for prices. That is

$$p_{m,i} = p_i(n_{F,i}) = p_i[(n_i - m_i) \tau^{1-\sigma} + m_i].$$

We next use equations (6), (8), (9), and (10) to calculate the response of p_i and $p_{m,i}$ to changes in the number of product lines, n_i , changes in marginal cost, a_i , changes in the product span of foreign subsidiaries, m_i , and changes in τ and the price index P . Denoting by a hat the proportional rate of change of a variable, i.e., $\hat{x} = dx/x$, the fact that $p_{e,i} = \tau p_{m,i}$ and the definition $n_{F,i} := (n_i - m_i) \tau^{1-\sigma} + m_i$, differentiating these equations yields the solutions:

$$\hat{p}_i = \frac{\beta_i}{1 + (\sigma - 1)\beta_i} \hat{n}_i + \frac{1}{1 + (\sigma - 1)\beta_i} \hat{a}_i + \frac{(\sigma - 1)\beta_i}{1 + (\sigma - 1)\beta_i} \hat{P}, \quad (32)$$

$$\hat{p}_{m,i} = \frac{\beta_{F,i}}{1 + (\sigma - 1)\beta_{F,i}} \hat{n}_{F,i} + \frac{1}{1 + (\sigma - 1)\beta_{F,i}} \hat{a}_i + \frac{\beta_{F,i}(\sigma - 1)}{1 + (\sigma - 1)\beta_{F,i}} \hat{P},$$

where

$$\beta_i = \frac{\delta s_i}{(\sigma - \delta s_i - 1)(\sigma - \delta s_i)} > 0, \quad (33)$$

$$\beta_{F,i} = \frac{\delta s_{F,i}}{(\sigma - \delta s_{F,i} - 1)(\sigma - \delta s_{F,i})} > 0. \quad (34)$$

Using the definition $n_{F,i} = (n_i - m_i) \tau^{1-\sigma} + m_i$, the equation for $\hat{p}_{m,i}$ then becomes

$$\begin{aligned} \hat{p}_{m,i} = & \frac{\beta_{F,i}}{1 + (\sigma - 1)\beta_{F,i}} \frac{n_i \tau^{1-\sigma}}{(n_i - m_i) \tau^{1-\sigma} + m_i} \hat{n}_i + \frac{1}{1 + (\sigma - 1)\beta_{F,i}} \hat{a}_i \\ & + \frac{\beta_{F,i}}{1 + (\sigma - 1)\beta_{F,i}} \frac{(1 - \tau^{1-\sigma}) m_i}{(n_i - m_i) \tau^{1-\sigma} + m_i} \hat{m}_i, \\ & - \frac{\beta_{F,i}(\sigma - 1)}{1 + (\sigma - 1)\beta_{F,i}} \frac{(n_i - m_i) \tau^{1-\sigma}}{(n_i - m_i) \tau^{1-\sigma} + m_i} \hat{\tau} + \frac{\beta_{F,i}(\sigma - 1)}{1 + (\sigma - 1)\beta_{F,i}} \hat{P}. \end{aligned} \quad (35)$$

C Comparative Dynamics of FDI

First, we derive the expression shown in equation (17). We begin from the first order condition expressed in the following form:

$$-\frac{\partial \mathcal{H}}{\partial m_i} = -\frac{\partial \pi_{F,i}}{\partial m_i} + \zeta_{m,i} \theta = \dot{\zeta}_{m,i} - r \zeta_{m,i}.$$

We proceed by evaluating the marginal profits from increasing the span of foreign subsidiaries: $\frac{\partial \pi_{F,i}}{\partial m_i}$.

By substituting in equation (16), this term becomes:

$$\frac{\partial \pi_{F,i}}{\partial m_i} = \frac{\partial P^\delta [(n_i - m_i) \tau^{1-\sigma} + m_i] p_{m,i} (n_i, m_i)^{-\sigma} [p_{m,i} (n_i, m_i) - a_i] - \iota_{m,i}}{\partial m_i}.$$

Evaluating this derivative with respect to m_i results in

$$\begin{aligned} \frac{\partial \pi_{F,i}}{\partial m_i} = & P^\delta p_{m,i}^{-\sigma} [1 - \tau^{1-\sigma}] (p_i - a_i) \\ & + P^\delta p_{m,i}^{-\sigma} [(n_i - m_i) \tau^{1-\sigma} + m_i] \left[(1 - \sigma) + \sigma a_i p_{m,i}^{-1} \right] \frac{\partial p_{m,i}}{\partial m_i}. \end{aligned} \quad (36)$$

From equation (35) we know that the elasticity of foreign subsidiary price with respect to the foreign subsidiary product span is given by

$$\frac{\partial p_{m,i}}{\partial m_i} \frac{m_i}{p_{m,i}} = \frac{\beta_{F,i}}{1 + (\sigma - 1) \beta_{F,i}} \frac{(1 - \tau^{1-\sigma}) m_i}{(n_i - m_i) \tau^{1-\sigma} + m_i}.$$

Substituting this into equation (36) yields

$$\begin{aligned} \frac{1}{(1 - \tau^{1-\sigma})} \frac{\partial \pi_{F,i}}{\partial m_i} = & P^\delta p_{m,i}^{-\sigma} (p_i - a_i) \\ & + P^\delta p_{m,i}^{1-\sigma} \left[(1 - \sigma) + \sigma a_i p_{m,i}^{-1} \right] \frac{\beta_{F,i}}{1 + (\sigma - 1) \beta_{F,i}}. \end{aligned}$$

Substituting in equation (34) and simplifying returns the desired result:

$$\frac{\partial \pi_{F,i}}{\partial m_i} = (1 - \tau^{1-\sigma}) \Lambda_i [s_{F,i} (n_i, m_i)],$$

where

$$\Lambda_i(s) = \left(\frac{\sigma - \delta s}{\sigma - \delta s - 1} \right)^{-\sigma} \frac{\sigma a_i^{1-\sigma} P^\delta}{\sigma(\sigma - \delta s - 1) + (\delta s)^2}. \quad (37)$$

In order to understand how parameter changes affect firm dynamics of foreign direct investment, it is sufficient to understand how they affect the steady state equations. A change in labor productivity of firm i , $1/a_i$, affects the steady state equation (19) but not equation (20). The effect of

improved labor productivity on the firm's dynamics arises from the increased profitability of FDI. As discussed in the main body of the paper, there is a direct positive effect because each product produced by a subsidiary becomes more profitable, but there is also an indirect negative effect coming from the increase in the firm's market share. To see which of these dominates, it is sufficient to evaluate how the marginal profitability of expanding product span is affected by a change in productivity, i.e., $\frac{\partial \Lambda_i(s)}{\partial a_i} \frac{a_i}{\Lambda_i(s)}$. Totally differentiating (37) yields

$$\hat{\Lambda}_i = -(\sigma - 1)\hat{a}_i + \delta\hat{P} - \frac{\sigma\delta s}{(\sigma - \delta s - 1)(\sigma - \delta s)}\hat{s} + \frac{\delta s(\sigma - 2\delta s)}{(\sigma - \delta s - 1)\sigma + \delta^2 s^2}\hat{s}. \quad (38)$$

After collecting terms, we have

$$\hat{\Lambda}_i = -(\sigma - 1)\hat{a}_i + \delta\hat{P} - \delta^2 s^2 \frac{2(\sigma - \delta s - 1)(\sigma - \delta s) + \sigma(\sigma - 1)}{(\sigma - \delta s)(\sigma - \delta s - 1) \left[(\sigma - \delta s - 1)\sigma + (\delta s)^2 \right]} \hat{s}, \quad (39)$$

which also proves that $\Lambda_i(s)$ is a decreasing function of s . Next we can calculate the relationship between a_i and s , following the same derivation as for equation (32), to obtain

$$\hat{s} = -\frac{\sigma - 1}{1 + (\sigma - 1)\beta}\hat{a}_i. \quad (40)$$

Combining equations (38) and (40) yields

$$\frac{\partial \Lambda_i(s)}{\partial a_i} \frac{a_i}{\Lambda_i(s)} = \frac{(\sigma - 1)(\delta s)^2 - (\sigma - \delta s - 1)^2 \left[\sigma^2 - (\delta s)^2 \right]}{(\sigma - \delta s - 1)(\sigma - \delta s) \left[(\sigma - \delta s - 1)\sigma + \delta^2 s^2 \right]^2} (\sigma - 1). \quad (41)$$

When

$$(\sigma - 1)(\delta s_{F,i})^2 - (\sigma - \delta s_{F,i} - 1)^2 \left[\sigma^2 - (\delta s_{F,i})^2 \right] < 0, \quad (42)$$

an increase in labor productivity (decrease in a_i) shifts upward the $\dot{\zeta}_{m,i} = 0$ curve, leading to higher m_i in the long run. When this condition is reversed, an increase in labor productivity has an opposite effect on m_i .

This analysis shows that a firm's response to an improvement in labor productivity depends on its initial market share. Condition (42) is satisfied when the foreign market share is small. Moreover, since the left-hand side of this condition is increasing in $s_{F,i}$, it is always satisfied if it is satisfied for $s_{F,i} = 1$, i.e., if

$$(\varepsilon - 1)^2 \left[\sigma^2 - (\sigma - \varepsilon)^2 \right] > (\sigma - 1)(\sigma - \varepsilon)^2.$$

Clearly, the last inequality holds for $\sigma \rightarrow \varepsilon > 1$ and it is violated for $\varepsilon \rightarrow 1$. That being the case, there exists an s_c that satisfies

$$(\sigma - \delta s_c - 1)^2 \left[\sigma^2 - (\delta s_c)^2 \right] = (\sigma - 1)(\delta s_c)^2$$

such that the $\dot{\zeta}_{m,i} = 0$ curve shifts upward in response to a labor productivity improvement in firm i when $s_{F,i} < s_c$ and downward when $s_{F,i} > s_c$.

We now discuss entry of single-product firms. Since the price indexes $P_H = P_F = P$ are constant during the transition, equations (4) and (5) imply that so are $n_F p^{1-\sigma} + \sum_{k=1}^I n_{F,k} p_{m,k}^{1-\sigma}$ and $n_H p^{1-\sigma} + \sum_{k=1}^I n_k p_k^{1-\sigma}$, where $n_{F,k} = (n_k - m_k) \tau^{1-\sigma} + m_k$. In the home country prices do not change. Therefore the number of the home country single-product firms, n_H , remains constant on the transition path. In the foreign country the effective number of products of firm i , $n_{F,i}$, rises when $m_i(0)$ is below its steady value, and so does its foreign price $p_{m,k}$. Therefore the number of single product firms declines over time if and only if the product $n_{F,i} p_{m,i}^{1-\sigma}$ rises. Now recall that $p_{m,i}$ is an increasing function of $n_{F,i}$, $p_{m,i} = p_i(n_{F,i})$ (see (13)), and we show in the appendix that the elasticity of the function $p_i(n_{F,i})$ is $\beta_{F,i} / [1 + (\sigma - 1)\beta_{F,i}]$, where $\beta_{F,i} > 0$.³⁹ It follows that the product $n_{F,i} p_{m,i}^{1-\sigma}$ rises during this transition and the number of foreign single product firms, n_F , declines. Naturally, when $m_i(0)$ is above its steady value the number of single-product firms does not change in the home country and the number of single-product firms rises in the foreign country.

D Merger

We now consider a merger between two larger multinationals. Without loss of generality, suppose that firms $i = 1, 2$ merge. This merger does not change the business strategy of the other large firms and the price indexes at home and abroad still satisfy (4) and (5).

Assume that the combined firm has to produce every product in span n_i with the technology a_i , and we focus on the case in which these product spans are exogenous and fixed. Then in the instantaneous equilibrium, in which the scope of foreign product lines is given, the decision problem of the merged firm is to maximize joint profits after entry of single-product firms. Namely, the merged form solves the following problem:

$$\max_{\{p_i, p_{e,i}, p_{m,i}\}_{i=1,2}} \sum_{i=1}^2 n_i P_H^\delta p_i^{-\sigma} (p_i - a_i) + \sum_{i=1}^2 (n_i - m_i) P_F^\delta p_{e,i}^{-\sigma} (p_{e,i} - \tau a_i) + \sum_{i=1}^2 m_i P_F^\delta p_{m,i}^{-\sigma} (p_{m,i} - a_i)$$

subject to (4) and (5). The first-order conditions of this problem, for $i = 1, 2$, imply

$$\delta \frac{n_i p_i^{1-\sigma}}{P_H^{1-\sigma}} = \frac{n_i p_i^{-\sigma} (p_i - a_i)}{\sum_{k=1}^2 n_k p_k^{-\sigma} (p_k - a_k)} \left(\sigma - \frac{p_i}{p_i - a_i} \right), \quad (43)$$

$$\delta \frac{(n_i - m_i) p_{e,i}^{1-\sigma}}{P_F^{1-\sigma}} = \frac{(n_i - m_i) p_{e,i}^{-\sigma} (p_{e,i} - \tau a_i)}{\sum_{k=1}^2 (n_k - m_k) p_{e,k}^{-\sigma} (p_{e,k} - \tau a_k) + \sum_{k=1}^2 m_k p_{m,k}^{-\sigma} (p_{m,k} - a_k)} \left(\sigma - \frac{p_{e,i}}{p_{e,i} - \tau a_i} \right), \quad (44)$$

³⁹More precisely,

$$\beta_{F,i} = \frac{\delta s_{F,i}}{(\sigma - \delta s_{F,i} - 1)(\sigma - \delta s_{F,i})} > 0.$$

$$\delta \frac{m_i p_{m,i}^{1-\sigma}}{P_F^{1-\sigma}} = \frac{m_i p_{m,i}^{-\sigma} (p_{m,i} - a_i)}{\sum_{k=1}^2 (n_k - m_k) p_{e,k}^{-\sigma} (p_{e,k} - \tau a_k) + \sum_{k=1}^2 m_k p_{m,k}^{-\sigma} (p_{m,k} - a_k)} \left(\sigma - \frac{p_{m,i}}{p_{m,i} - a_i} \right). \quad (45)$$

First note that (43) implies

$$\delta \frac{1}{P_H^{1-\sigma}} = \frac{1}{\sum_{k=1}^2 n_k p_k^{-\sigma} (p_k - a_k)} \left(\sigma \frac{p_i - a_i}{p_i} - 1 \right),$$

$$\delta s_i = \frac{n_i p_i^{-\sigma} (p_i - a_i)}{\sum_{k=1}^2 n_k p_k^{-\sigma} (p_k - a_k)} \left(\sigma - \frac{p_i}{p_i - a_i} \right),$$

where s_i is the market share of products n_i in the home market. The first of these equations implies that p_i is proportional to a_i with the same factor of proportionality for $i = 1, 2$, say κ . That is, $p_i = \kappa a_i$ for $i = 1, 2$. In this case the second equation implies

$$\delta s_i = \frac{n_i a_i^{1-\sigma}}{\sum_{k=1}^2 n_k a_k^{-\sigma}} \left(\sigma - \frac{\kappa}{\kappa - 1} \right).$$

Adding up this equation across the merged firms yields

$$\delta s_{1+2} = \sigma - \frac{\kappa}{\kappa - 1} \Rightarrow \kappa = \frac{\sigma - \delta s_{1+2}}{\sigma - \delta s_{1+2} - 1},$$

where s_{1+2} is the merged firm's market share in the home country. It follows that home prices of products n_i are

$$p_i = \frac{\sigma - \delta s_{1+2}}{\sigma - \delta s_{1+2} - 1} a_i, \quad i = 1, 2. \quad (46)$$

Next consider (44) and (45). They imply

$$\delta \frac{1}{P_H^{1-\sigma}} = \frac{1}{\sum_{k=1}^2 (n_k - m_k) p_{e,k}^{-\sigma} (p_{e,k} - \tau a_k) + \sum_{k=1}^2 m_k p_{m,k}^{-\sigma} (p_{m,k} - a_k)} \left(\sigma \frac{p_{i,e} - \tau a_i}{p_{i,e}} - 1 \right),$$

$$\delta \frac{1}{P_H^{1-\sigma}} = \frac{1}{\sum_{k=1}^2 (n_k - m_k) p_{e,k}^{-\sigma} (p_{e,k} - \tau a_k) + \sum_{k=1}^2 m_k p_{m,k}^{-\sigma} (p_{m,k} - a_k)} \left(\sigma \frac{p_{i,m} - a_i}{p_{i,m}} - 1 \right),$$

$$\delta s_{i,e} = \frac{(n_i - m_i) p_{e,i}^{-\sigma} (p_{e,i} - \tau a_i)}{\sum_{k=1}^2 (n_k - m_k) p_{e,k}^{-\sigma} (p_{e,k} - \tau a_k) + \sum_{k=1}^2 m_k p_{m,k}^{-\sigma} (p_{m,k} - a_k)} \left(\sigma - \frac{p_{i,e}}{p_{i,e} - \tau a_i} \right),$$

$$\delta s_{i,m} = \frac{m_i p_{m,i}^{-\sigma} (p_{m,i} - a_i)}{\sum_{k=1}^2 (n_k - m_k) p_{e,k}^{-\sigma} (p_{e,k} - \tau a_k) + \sum_{k=1}^2 m_k p_{m,k}^{-\sigma} (p_{m,k} - a_k)} \left(\sigma - \frac{p_{i,m}}{p_{i,m} - a_i} \right),$$

where $s_{i,e}$ is the market share of exported products $n_i - m_i$ in the foreign country and $s_{i,m}$ is the

market share of foreign subsidiary sales of product lines m_i in the foreign market. The first two of these equations imply $p_{i,e} = \kappa_F \tau a_i = \tau p_{i,m}$ for some constant κ_F . In this event the last two equations imply

$$\delta s_{F,1+2} = \sigma - \frac{\kappa_F}{\kappa_F - 1} \Rightarrow \kappa_F = \frac{\sigma - \delta s_{F,1+2}}{\sigma - \delta s_{F,1+2} - 1},$$

where $s_{F,1+2} := \sum_{k=1}^2 (s_{i,e} + s_{i,m})$ is the foreign market share of the merged firm. Therefore

$$p_{e,i} = \frac{\sigma - \delta s_{F,1+2}}{\sigma - \delta s_{F,1+2} - 1} \tau a_i, \quad i = 1, 2, \quad (47)$$

$$p_{m,i} = \frac{\sigma - \delta s_{F,1+2}}{\sigma - \delta s_{F,1+2} - 1} a_i, \quad i = 1, 2. \quad (48)$$

The resulting market shares are

$$s_{1+2} = \frac{\sum_{i=1}^2 n_i p_i^{1-\sigma}}{P_H^{1-\sigma}}, \quad (49)$$

$$s_{F,1+2} = \frac{\sum_{i=1}^2 (n_i - m_i) p_{e,i}^{1-\sigma} + \sum_{i=1}^2 m_i p_{m,i}^{1-\sigma}}{P_F^{1-\sigma}}. \quad (50)$$

Equations (46) and (49) provide a solution to p_i , $i = 1, 2$ and s_{1+2} , while equations (47)-(48) and (50) provide a solution to $\{p_{i,e}, p_{i,m}\}$ for $i = 1, 2$ and for $s_{F,1+2}$. The combined market share at home is a function of both n_1 and n_2 while the combined foreign market share is a function of n_1 , n_2 , m_1 and m_2 , which we express as $s_{1+2}(n_1, n_2)$ and $s_{F,1+2}(n_1, n_2, m_1, m_2)$, respectively. Moreover, the domestic prices are functions of $s_{1+2}(n_1, n_2)$, $p_i[s_{1+2}(n_1, n_2)]$ for products n_i , where the function $p_i[\cdot]$ is the same as in the absence of a merger, while the prices charged by foreign affiliates for products m_i are a function of $s_{F,1+2}(n_1, n_2, m_1, m_2)$, expressed as

$$p_{i,m}(n_1, n_2, m_1, m_2) := p_i[s_{F,1+2}(n_1, n_2, m_1, m_2)].$$

Since the product spans n_1 and n_2 are constant, the merged firm adjusts prices in the home country instantly and keeps them at the new level for the rest of time. It also adjusts foreign prices instantly, both for exports and for subsidiary sales, expect that foreign prices change over time in response to foreign direct investment.

We can now use these characterizations of prices and market shares to formulate the merged firm's dynamic decision concerning foreign direct investment. The firm maximizes the discounted present value of operating profits minus FDI costs, expect than now it can increase the product span of goods produced with technology a_1 or a_2 . Due to the convex costs of FDI and the Inada conditions (i.e., the properties of $\phi(\iota)$), it engages in positive FDI in the vicinity of a steady state.⁴⁰

⁴⁰Far from the steady state it may be optimal to engage in foreign direct investment of m_1 or m_2 , but not both.

For this reason the decision problem in the vicinity of a steady state can be expressed as

$$\max_{\{\iota_{m,1}(t), m_1(t), \iota_{m,2}(t), m_2(t)\}_{t \geq 0}} \int_0^\infty e^{-rt} \pi_{F,1+2} [\iota_{m,1}(t), m_1(t), \iota_{m,2}(t), m_2(t)] dt$$

subject to (15), the initial conditions $m_i(0) = m_i^0$ for $i = 1, 2$ and a the transversality conditions stated below, where

$$\begin{aligned} \pi_{F,1+2}(\iota_{m,1}, m_1, \iota_{m,2}, m_2) = & \quad (51) \\ & P^\delta [(n_1 - m_1) \tau^{1-\sigma} + m_1] p_{m,1}(n_1, m_1, n_2, m_2)^{-\sigma} [p_{m,1}(n_1, m_1, n_2, m_2) - a_1] + \\ & P^\delta [(n_2 - m_2) \tau^{1-\sigma} + m_2] p_{m,2}(n_1, m_1, n_2, m_2)^{-\sigma} [p_{m,2}(n_1, m_1, n_2, m_2) - a_2] - \\ & (\iota_{m,1} + \iota_{m,2}), \text{ for all } t \geq 0. \end{aligned}$$

In this problem $\iota_{m,1}$ and $\iota_{m,2}$ are control variables while m_1 and m_2 are state variables.

The current-value Hamiltonian of this problem is

$$\mathcal{H}(\iota_{m,1}, m_1, \zeta_{m,1}, \iota_{m,2}, m_2, \zeta_{m,2}) = \pi_{F,1+2}(\iota_{m,1}, m_1, \iota_{m,2}, m_2) + \sum_{i=1}^2 \zeta_{m,i} [\phi_m(\iota_{m,i}) - \theta m_i],$$

where $\zeta_{m,1}$ and $\zeta_{m,2}$ are the co-state variables of (15). Assuming an interior solution, the first-order conditions of this optimal control problem are

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \iota_{m,i}} &= \zeta_{m,i} \phi'_m(\iota_{m,i}) - 1 = 0, \quad i = 1, 2, \\ -\frac{\partial \mathcal{H}}{\partial m_i} &= -\frac{\partial \pi_{F,1+2}}{\partial m_i} + \zeta_{m,i} \theta = \dot{\zeta}_{m,i} - r \zeta_{m,i}, \quad i = 1, 2, \end{aligned}$$

and the transversality conditions are

$$\lim_{t \rightarrow \infty} e^{-rt} \zeta_{m,i}(t) m_i(t) = 0, \quad i = 1, 2.$$

Since

$$\frac{\partial \pi_{F,1+2}}{\partial m_i} = (1 - \tau^{1-\sigma}) \Lambda_i[s_{F,1+2}(n_1, m_1, n_2, m_2)], \quad i = 1, 2,$$

where, as before,

$$\Lambda_i(s) := \left(\frac{\sigma - \delta s}{\sigma - \delta s - 1} \right)^{-\sigma} \frac{a_i^{1-\sigma} P^\delta \sigma}{(\sigma - \delta s - 1) \sigma + (s\delta)^2}, \quad \Lambda'_i(s) < 0,$$

and the steady state conditions can be expressed as

$$(r + \theta) \zeta_{m,i} = (1 - \tau^{1-\sigma}) \Lambda_i[s_{F,1+2}(n_1, m_1, n_2, m_2)], \quad i = 1, 2, \quad (52)$$

$$\phi_m \left[\varphi_m \left(\zeta_{m,i}^{-1} \right) \right] = \theta m_i, \quad i = 1, 2. \quad (53)$$

The merged firm has a larger foreign market share than either firm 1 or 2 had before the merger.⁴¹ As a result, (52) implies that the value of a marginal product line of m_i , $\zeta_{m,i}$, is lower in the steady state of the merged firm than it was before the merger, for $i = 1, 2$. That being the case, (53) implies that the merged firm engages in less FDI and has a lower product span m_i in steady state. This is yet another manifestation of the substitutability between the total product span and the product spans of foreign affiliates.

E Full Dynamic Problem

As described in the main text, firm i solves an optimal control problem in which m_i and n_i are state variables while FDI, $\iota_{m,i}$, and investment in innovation, $\iota_{n,i}$, are control variables, facing the initial conditions $m_i(0) = m_i^0$ and $n_i(0) = n_i^0$. The two governing equations for the evolution of product span of foreign subsidiaries and the firm's product span are given by equations (15) and (25), respectively, reproduced below:

$$\dot{m}_i = \phi_m(\iota_{m,i}) - \theta m_i, \text{ for all } t \geq 0, \quad (54)$$

$$\dot{n}_i = \phi_n(\iota_{n,i}) - \theta n_i, \text{ for all } t \geq 0. \quad (55)$$

At every point in time the firms play a three stage game. In the first stage multi-product firms invest in innovation and in foreign subsidiaries. Namely, they choose $\iota_{n,i}$ and $\iota_{m,i}$. In the second stage single-product firms enter and they live only one instant of time. For this reason, they make profits only in this single instant. In the third stage all firms choose prices, in the manner described in Section 3. Under the circumstances in a subgame perfect equilibrium of the first stage game the price index P is expected to be determined by the free entry condition (12), and it remains constant as long as the cost of entry and the cost of production of the single-product firms do no change. It follows that the profit flow of large firm i is:⁴²

$$\begin{aligned} \pi_i(\iota_{n,i}, \iota_{m,i}, n_i, m_i) &:= P^\delta n_i p_i(n_i)^{-\sigma} [p_i(n_i) - a_i] \\ &+ P^\delta n_i p_i(n_i)^{-\sigma} [p_i(n_i) - a_i] \\ &+ P^\delta [(n_i - m_i) \tau^{1-\sigma} + m_i] p_{m,i}(n_i, m_i)^{-\sigma} [p_{m,i}(n_i, m_i) - a_i] - \iota_{n,i} - \iota_{m,i}, \\ &\text{for all } t \geq 0, \end{aligned} \quad (56)$$

$$p_{m,i}(n_i, m_i) := p_i[(n_i - m_i) \tau^{1-\sigma} + m_i],$$

⁴¹For suppose to the contrary, $s_{F,1+2}$ is smaller than either $s_{F,1}$ or $s_{F,2}$ were before the merger. In this case the steady state conditions imply that both m_1 and m_2 are larger after the merger than they were before the merger, which contradicts the supposition that after the merger $s_{F,1+2}$ is smaller than either $s_{F,1}$ or $s_{F,2}$ were before the merger.

⁴²Recall that the export price of a variety not produced in F is $p_{e,i} = \tau p_{m,i}$.

where P is the same at every t while π_i , n_i , m_i , p_i , $p_{m,i}$, $\iota_{n,i}$ and $\iota_{m,i}$ vary over time. Moreover, p_i is given by (6) and $p_{m,i}$ is given by (8). In this economy the state vector is $\{\mathbf{n}, \mathbf{m}\} = \{(n_1, n_2, \dots, n_I), (m_1, m_2, \dots, m_I)\}$, a function of time t , and the prices p_i and $p_{m,i}$ vary over time as functions of $\mathbf{n}, \mathbf{m}$. Note, however, from (6) and (8) that as long as all the parameters remain constant (and therefore $P_H = P_F = P$ remain constant as well) p_i and $p_{m,i}$ depend only on the elements $\{n_i, m_i\}$ of $\{\mathbf{n}, \mathbf{m}\}$ on the dynamic path.

We now use (6), (8), (9) and (10) to obtain the elasticities of the functions $p_i(n_i)$, $s_i(n_i)$, $p_{m,i}(n_i, m_i) := p_i[(n_i - m_i)\tau^{1-\sigma} + m_i]$ and $s_{F,i}(n_i, m_i) := s_i[(n_i - m_i)\tau^{1-\sigma} + m_i]$:

$$\frac{\partial p_i}{\partial n_i} \frac{n_i}{p_i} = \frac{\beta_i}{1 + (\sigma - 1)\beta_i}, \quad (57)$$

$$\frac{\partial s_i}{\partial n_i} \frac{n_i}{s_i} = \frac{1}{1 + (\sigma - 1)\beta_i}, \quad (58)$$

$$\frac{\partial p_{m,i}}{\partial n_i} \frac{n_i}{p_{m,i}} = \frac{\beta_{F,i}}{1 + (\sigma - 1)\beta_{F,i}} \frac{n_i \tau^{1-\sigma}}{(n_i - m_i) \tau^{1-\sigma} + m_i}, \quad (59)$$

$$\frac{\partial s_{F,i}}{\partial n_i} \frac{n_i}{s_{F,i}} = \frac{1}{1 + (\sigma - 1)\beta_{F,i}} \frac{n_i \tau^{1-\sigma}}{(n_i - m_i) \tau^{1-\sigma} + m_i}, \quad (60)$$

$$\frac{\partial p_{m,i}}{\partial m_i} \frac{m_i}{p_{m,i}} = \frac{\beta_{F,i}}{1 + (\sigma - 1)\beta_{F,i}} \frac{(1 - \tau^{1-\sigma}) m_i}{(n_i - m_i) \tau^{1-\sigma} + m_i}, \quad (61)$$

$$\frac{\partial s_{F,i}}{\partial m_i} \frac{m_i}{s_{F,i}} = \frac{1}{1 + (\sigma - 1)\beta_{F,i}} \frac{(1 - \tau^{1-\sigma}) m_i}{(n_i - m_i) \tau^{1-\sigma} + m_i}, \quad (62)$$

where β_i and $\beta_{F,i}$ are defined in equations (33) and (34). Note that $\beta_{F,i}$ is increasing in $s_{F,i}$ and $s_{F,i}$ is increasing in n_i ; therefore $\beta_{F,i}$ is increasing in n_i . The analogous statements are true for β_i , s_i and n_i . As a result, the elasticity of the domestic price function is larger the larger is s_i while the elasticity of the domestic market share function is smaller the larger is s_i , and similarly for the foreign price elasticity and foreign market share.

Next assume that the interest rate is constant and equal to r . This interest rate can be derived from the assumption that individuals discount future utility flows (1) with a constant rate r , so that they maximize the discounted present value of utility $\int_0^\infty e^{-rt} u(t) dt$. Under these circumstances firm i maximizes the discounted present value of its profits net of investment costs. It therefore solves the following optimal control problem:

$$\max_{\{\iota_{n,i}(t), \iota_{m,i}(t), n_i(t), m_i(t)\}_{t \geq 0}} \int_0^\infty e^{-rt} \pi_i[\iota_{n,i}(t), \iota_{m,i}(t), n_i(t), m_i(t)] dt$$

subject to (55), (54), (56), $n_i(0) = n_i^0$, $m_i(0) = m_i^0$, and two transversality conditions to be described below. In this problem $\iota_{n,i}(t)$ and $\iota_{m,i}(t)$ are control variables while $n_i(t)$ and $m_i(t)$ are state variables.

The current-value Hamiltonian of this problem is:

$$\begin{aligned}\mathcal{H}(\iota_{n,i}, \iota_{m,i}, n_i, m_i, \zeta_{n,i}, \zeta_{m,i}) &= \pi_i(\iota_{n,i}, \iota_{m,i}, n_i, m_i) \\ &+ \zeta_{n,i} [\phi_n(\iota_{n,i}) - \theta n_i] + \zeta_{m,i} [\phi_m(\iota_{m,i}) - \theta m_i] - \iota_{n,i} - \iota_{m,i},\end{aligned}$$

where $\pi_i(\iota_{n,i}, \iota_{m,i}, n_i, m_i)$ is given in (56), $\zeta_{n,i}$ is the co-state variable of constraint (55) and $\zeta_{m,i}$ is the co-state variable of (54). These co-state variables vary over time. The first-order conditions of this optimal control problem are:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \iota_{n,i}} &= -1 + \zeta_{n,i} \phi'_n(\iota_{n,i}) = 0, \\ \frac{\partial \mathcal{H}}{\partial \iota_{m,i}} &= -1 + \zeta_{m,i} \phi'_m(\iota_{m,i}) = 0, \\ -\frac{\partial \mathcal{H}}{\partial n_i} &= -\frac{\partial \pi_i(\iota_{n,i}, \iota_{m,i}, n_i, m_i)}{\partial n_i} + \theta \zeta_{n,i} = \dot{\zeta}_{n,i} - r \zeta_{n,i}, \\ -\frac{\partial \mathcal{H}}{\partial m_i} &= -\frac{\partial \pi_i(\iota_{n,i}, \iota_{m,i}, n_i, m_i)}{\partial m_i} + \zeta_{m,i} \theta = \dot{\zeta}_{m,i} - r \zeta_{m,i},\end{aligned}$$

and the transversality conditions are

$$\begin{aligned}\lim_{t \rightarrow \infty} e^{-rt} \zeta_{n,i}(t) n_i(t) &= 0, \\ \lim_{t \rightarrow \infty} e^{-rt} \zeta_{m,i}(t) m_i(t) &= 0.\end{aligned}$$

In addition, the optimal path of $(\iota_{n,i}, \iota_{m,i}, n_i, m_i)$ has to satisfy the differential equations (55) and (54), starting at n_i^0 and m_i^0 .

Now note that

$$\frac{\partial \pi_i}{\partial n_i} = \frac{\partial P^\delta \{n_i p_i(n_i)^{-\sigma} [p_i(n_i) - a_i] + [(n_i - m_i) \tau^{1-\sigma} + m_i] p_{m,i}(n_i, m_i)^{-\sigma} (p_{m,i}(n_i, m_i) - a_i)\}}{\partial n_i}.$$

We can evaluate the marginal profitability of increased product span separately for the home and foreign markets. Consider first the marginal productivity in the home market:

$$\frac{\partial P^\delta n_i p_i(n_i)^{-\sigma} [p_i(n_i) - a_i]}{\partial n_i} = P^\delta p_i^{-\sigma} \left[(p_i - a_i) + p_i [(1 - \sigma) + \sigma a_i p_i^{-1}] \frac{\partial p_i}{\partial n_i} \frac{n_i}{p_i} \right].$$

Substituting in equation (57) and (6), we get that

$$\begin{aligned}\frac{\partial P^\delta n_i p_i (n_i)^{-\sigma} (p_i(n_i) - a_i)}{\partial n_i} &= P^\delta \left(\frac{\sigma - \delta s_i}{\sigma - \delta s_i - 1} \right)^{-\sigma} a_i^{1-\sigma} \left[\frac{1}{\sigma - \delta s_i - 1} - \frac{1}{\sigma - \delta s_i - 1} \frac{\delta^2 s_i^2}{\sigma(\sigma - \delta s_i - 1) + (\delta s_i)^2} \right] \\ &= P^\delta \left(\frac{\sigma - \delta s_i}{\sigma - \delta s_i - 1} \right)^{-\sigma} a_i^{1-\sigma} \frac{\sigma}{\sigma(\sigma - \delta s_i - 1) + (\delta s_i)^2}.\end{aligned}$$

Due to the relationship between $p_{m,i}$ and p_i shown in equation (13), these results are sufficient to solve for the total marginal profitability of increasing product span:

$$\frac{\partial \pi_i (\iota_{n,i}, \iota_{m,i}, n_i, m_i)}{\partial n_i} = \pi_{i,n} (n_i, m_i) := \Lambda_i [s_i (n_i)] + \tau^{1-\sigma} \Lambda_i [s_{F,i} (n_i, m_i)],$$

where $\pi_{i,n} (n_i, m_i)$ is the marginal profit of n_i . We can solve analogously for an increase in the foreign subsidiary product span, which was also solve for above in section (C). As before we get

$$\frac{\partial \pi_i (\iota_{n,i}, \iota_{m,i}, n_i, m_i)}{\partial m_i} = \pi_{i,m} (n_i, m_i) \equiv (1 - \tau^{1-\sigma}) \Lambda_i [s_{F,i} (n_i, m_i)]$$

where we define $\pi_{i,m} (n_i, m_i)$ as the marginal profit of m_i . Therefore $\pi_{i,n} (n_i, m_i)$ is declining in n_i while $\pi_{i,m} (n_i, m_i)$ is declining in m_i .

The above first-order conditions can be expressed as:

$$\zeta_{n,i} \phi'_n (\iota_{n,i}) = 1, \quad (63)$$

$$\zeta_{m,i} \phi'_m (\iota_{m,i}) = 1, \quad (64)$$

$$\dot{\zeta}_{n,i} = (r + \theta) \zeta_{n,i} - \Lambda_i [s_i (n_i)] - \tau^{1-\sigma} \Lambda_i [s_{F,i} (n_i, m_i)], \quad (65)$$

$$\dot{\zeta}_{m,i} = (r + \theta) \zeta_{m,i} - (1 - \tau^{1-\sigma}) \Lambda_i [s_{F,i} (n_i, m_i)]. \quad (66)$$

We can now use (63) and (64) to solve the investment levels $\iota_{n,i}$ and $\iota_{m,i}$,

$$\iota_{n,i} = \varphi_n \left(\zeta_{n,i}^{-1} \right),$$

$$\iota_{m,i} = \varphi_m \left(\zeta_{m,i}^{-1} \right),$$

where $\varphi_n (\cdot)$ is the inverse of $\phi'_n (\cdot)$ and $\varphi_m (\cdot)$ is the inverse of $\phi'_m (\cdot)$. Substituting these functions into (55), (54), (65) and (66) yields an autonomous system of four differential equations with two initial conditions, n_i^0 and m_i^0 and free choices of $\zeta_i (0)$ and $\zeta_{m,i} (0)$ shown below:

$$\dot{\zeta}_{m,i} = (r + \theta) \zeta_{m,i} - (1 - \tau^{1-\sigma}) \Lambda_i \left\{ s_i [(n_i - m_i) \tau^{1-\sigma} + m_i] \right\},$$

$$\dot{m}_i = \phi_m \left[\varphi_m \left(\zeta_{m,i}^{-1} \right) \right] - \theta m_i,$$

$$\dot{\zeta}_{n,i} = (r + \theta) \zeta_{n,i} - \Lambda_i [s_i (n_i)] - \tau^{1-\sigma} \Lambda_i \left\{ s_i [(n_i - m_i) \tau^{1-\sigma} + m_i] \right\},$$

$$\dot{n}_i = \phi_n \left[\varphi_n \left(\zeta_{n,i}^{-1} \right) \right] - \theta n_i.$$

F Saddle Path Stability

The four differential equations derived at the end of the previous section satisfy local saddle-path stability if its linearized representation around a steady state has two positive and two negative characteristic roots, because the dynamic system has two state variables and two jump variables. Let the steady state values be $\tilde{\zeta}_{m,i}$, $\tilde{\zeta}_{n,i}$, $\tilde{m}_i$, $\tilde{n}_i$, $\tilde{s}_{F,i}$ and $\tilde{s}_i$. Then the linearized system is

$$\dot{\mathbf{y}} = A (\mathbf{y} - \tilde{\mathbf{y}}),$$

where

$$\mathbf{y} = \begin{pmatrix} \zeta_{m,i} \\ m_i \\ \zeta_{n,i} \\ n_i \end{pmatrix},$$

$\tilde{\mathbf{y}}$ is the steady state value of $\mathbf{y}$ and A is the matrix

$$A = \begin{pmatrix} r + \theta & a_{12} & 0 & a_{14} \\ a_{21} & -\theta & 0 & 0 \\ 0 & a_{32} & r + \theta & a_{34} \\ 0 & 0 & a_{43} & -\theta \end{pmatrix},$$

$$a_{12} = - (1 - \tau^{1-\sigma})^2 \Lambda'_i (\tilde{s}_{F,i}) s'_i [(\tilde{n}_i - \tilde{m}_i) \tau^{1-\sigma} + \tilde{m}_i],$$

$$a_{14} = - (1 - \tau^{1-\sigma}) \tau^{1-\sigma} \Lambda'_i (\tilde{s}_{F,i}) s'_i [(\tilde{n}_i - \tilde{m}_i) \tau^{1-\sigma} + \tilde{m}_i],$$

$$a_{21} = -\phi'_m \left[\varphi_m \left(\tilde{\zeta}_{m,i}^{-1} \right) \right] \varphi'_m \left(\tilde{\zeta}_{m,i}^{-1} \right) \tilde{\zeta}_{m,i}^{-2},$$

$$a_{32} = a_{14},$$

$$a_{34} = -\Lambda'_i (\tilde{s}_i) s'_i (\tilde{n}_i) - \tau^{2(1-\sigma)} \Lambda'_i (\tilde{s}_{F,i}) s'_i [(\tilde{n}_i - \tilde{m}_i) \tau^{1-\sigma} + \tilde{m}_i],$$

$$a_{43} = -\phi'_n \left[\varphi_n \left(\tilde{\zeta}_{n,i}^{-1} \right) \right] \varphi'_n \left(\tilde{\zeta}_{n,i}^{-1} \right) \tilde{\zeta}_{n,i}^{-2}.$$

Note that all the a_{ij} s are positive.

Every characteristic roots λ of the matrix A satisfies

$$\det \begin{pmatrix} r + \theta - \lambda & a_{12} & 0 & a_{14} \\ a_{21} & -\theta - \lambda & 0 & 0 \\ 0 & a_{32} & r + \theta - \lambda & a_{34} \\ 0 & 0 & a_{43} & -\theta - \lambda \end{pmatrix} = 0,$$

or

$$q^2 + bq + c = 0,$$

where

$$q := (r + \theta - \lambda)(\theta + \lambda),$$

$$b := a_{34}a_{43} + a_{21}a_{12},$$

$$c := a_{21}a_{43}(a_{12}a_{34} - a_{14}a_{32}).$$

Note however that $b > 0$ and

$$a_{12}a_{34} - a_{14}a_{32} = (1 - \tau^{1-\sigma})^2 \Lambda'_i(\tilde{s}_{F,i}) s'_i[(\tilde{n}_i - \tilde{m}_i)\tau^{1-\sigma} + \tilde{m}_i] \Lambda'_i(\tilde{s}_i) s'_i(\tilde{n}_i) > 0.$$

Therefore $c > 0$ too. It therefore follows that

$$q = \frac{1}{2} \left(-b \pm \sqrt{b^2 - 4c} \right) < 0.$$

This yields two real solutions to q , $q_1 < 0$ and $q_2 < 0$, because $b^2 > 4c$. To prove the latter, not that $b^2 > 4c$ if and only if

$$a_{34}^2 a_{43}^2 + 2a_{21}a_{12}a_{34}a_{43} + a_{21}^2 a_{12}^2 + 4a_{21}a_{43}a_{14}a_{32} > 4a_{21}a_{12}a_{34}a_{43}.$$

This expression can, however, be rewritten as

$$(a_{21}a_{12} - a_{34}a_{43})^2 + 4a_{21}a_{43}a_{14}a_{32} > 0,$$

which is always satisfied. It follows that λ_{i1} and λ_{i2} that solve

$$(r + \theta - \lambda)(\theta + \lambda) = -\lambda^2 + r\lambda + \theta(r + \theta) = q_i$$

have opposite signs for every $i = 1, 2$. That is, for every q_i one characteristic root is positive and the other is negative. We conclude that the matrix A has four characteristic roots, two negative and two positive. Therefore, our dynamic system is locally saddle-path stable.

G Comparative Dynamics of n_i and m_i

We start with the slopes of the MM and NN curves. The absolute value of the slope of the MM curve is c_{mm}/c_{mn} while the absolute value of the slope of the NN curve is c_{nm}/c_{nn} , where $c_{nm} = c_{mn}$ and

$$c_{mm} = (r + \theta) \psi'_{m,i}(m_i) - (1 - \tau^{1-\sigma})^2 \Lambda'_i(s_{F,i}) s'_i(n_{F,i}),$$

$$c_{mn} = -(1 - \tau^{1-\sigma}) \tau^{1-\sigma} \Lambda'_i(s_{F,i}) s'_i(n_{F,i}),$$

$$c_{nn} = (r + \theta) \psi'_{n,i}(n_i) - \Lambda'_i(s_i) s'_i(n_i) - (\tau^{1-\sigma})^2 \Lambda'_i(s_{F,i}) s'_i(n_{F,i}).$$

It follows that the MM curve is steeper.

We next examine how a change in τ affects investments, concentrating on the induced shifts of the MM and NN curves. Differentiating equation (30), we confirm that an increase in τ shifts up the MM curve:

$$\frac{\partial (1 - \tau^{1-\sigma}) \Lambda_i \{s_i [(n_i - m_i) \tau^{1-\sigma} + m_i]\}}{\partial \tau} = (\sigma - 1) \tau^{-\sigma} \Lambda_i + (1 - \tau^{1-\sigma}) (n_i - m_i) \underbrace{\frac{\partial \Lambda_i}{\partial s_{F,i}}}_{<0} \underbrace{\frac{\partial s_{F,i}}{\partial \tau}}_{<0} > 0. \quad (67)$$

This result and (30), i.e.,

$$(r + \theta) \psi_{m,i}(m_i) = (1 - \tau^{1-\sigma}) \Lambda_i \{s_i [(n_i - m_i) \tau^{1-\sigma} + m_i]\},$$

imply that the MM curve shifts up in response to a rise in trade costs. It follows that the foreign market share must be higher on the new MM curve for a given level of m_i . In other words, the MM curve shifts so that at every level of m_i there is a higher value of n_i .

We can perform a similar analysis for the NN curve,

$$\begin{aligned} \frac{\partial (\Lambda_i [s_i (n_i)] + \tau^{1-\sigma} \Lambda_i \{s_i [(n_i - m_i) \tau^{1-\sigma} + m_i]\})}{\partial \tau} &= \frac{\partial \tau^{1-\sigma} \Lambda_i \{s_i [(n_i - m_i) \tau^{1-\sigma} + m_i]\}}{\partial \tau} \quad (68) \\ &= -(\sigma - 1) \tau^{-\sigma} \Lambda_i (s_{F,i}) + \tau^{1-\sigma} (n_i - m_i) \frac{\partial \Lambda_i}{\partial s_{F,i}} \frac{\partial s_{F,i}}{\partial n_{F,i}} \frac{\partial [(n_i - m_i) \tau^{1-\sigma} + m_i]}{\partial \tau}. \end{aligned}$$

Rearranging:

$$\begin{aligned} \frac{\partial \tau^{1-\sigma} \Lambda_i [s_i (n_{F,i})]}{\partial \tau} \frac{\tau^\sigma / (\sigma - 1)}{\Lambda_i [s_i (n_{F,i})]} &= -1 + \tau \frac{(n_i - m_i) / (\sigma - 1)}{\Lambda_i [s_i (n_{F,i})]} \frac{\partial \Lambda_i [s_i (n_{F,i})]}{\partial s_{F,i}} \frac{\partial s_{F,i}}{\partial n_{F,i}} \frac{\partial [(n_i - m_i) \tau^{1-\sigma} + m_i]}{\partial \tau} \\ &= -1 + \eta_{\Lambda_i} (s_{F,i}) \eta_{s_i} (n_{F,i}) \frac{\tau^{1-\sigma} (n_i - m_i)}{(n_i - m_i) \tau^{1-\sigma} + m_i}, \quad (69) \end{aligned}$$

where $\eta_{\Lambda_i} (s_{F,i})$ is the absolute value of the elasticity of $\Lambda_i (\cdot)$ evaluated at $s_{F,i}$ and $\eta_{s_i} (n_{F,i})$ is the elasticity of $s_i (\cdot)$ evaluated at $n_{F,i}$, and they are given by

$$\begin{aligned} \eta_{s_i} (n) &= \frac{[\sigma - \delta s_i (n) - 1] [\sigma - \delta s_i (n)]}{[\sigma - \delta s_i (n) - 1] [\sigma - \delta s_i (n)] + (\sigma - 1) \delta s_i (n)}, \\ \eta_{\Lambda_i} (s) &= \delta^2 s^2 \frac{2(\sigma - \delta s - 1)(\sigma - \delta s) + \sigma(\sigma - 1)}{(\sigma - \delta s - 1)(\sigma - \delta s)[(\sigma - \delta s - 1)\sigma + \delta^2 s^2]}. \end{aligned}$$

If the right-hand side of (69) is positive, the NN curve shift out in response to an increase in τ . Otherwise it shifts in.

Proposition 4 shows that for a constant product span n_i an increase in τ raises firm i 's span of foreign subsidiaries and reduces exports relative to subsidiary sales. At the same time the firm makes no changes in the domestic market. Now, when product spans are endogenous, an increase in τ shifts rightward curve MM in Figure 3, as we have established above. If curve NN were to remain anchored at the original equilibrium point, this would imply that m_i rises in the long run and n_i declines. Yet NN is unlikely to remain put; it shifts down if and only if (see (69))

$$\eta_{A_i}(s_{F,i}) \eta_{s_i}(n_{F,i}) \frac{(n_i - m_i) \tau^{1-\sigma}}{(n_i - m_i) \tau^{1-\sigma} + m_i} < 1,$$

This condition is more likely to be satisfied when the foreign market share is small. When NN shifts down, the final outcome is higher m_i and lower n_i . But even if NN shifts up the final outcome may still be higher m_i and lower n_i . When m_i rises and n_i declines the firm manufactures a wider span of products in foreign subsidiaries and exports fewer products from home. As a result, exports decline relative to subsidiary sales. At the same time the firm sells fewer products in the home market, where it reduces prices, markups and its market share.

Next, consider changes in productivity. For a change in a_i , differentiation of the steady state equations (30) and (31) yields

$$c_{mm} dm_i + c_{mn} dn_i = (1 - \tau^{1-\sigma}) \tilde{A}_{i,a_i}(s_{F,i}) da_i,$$

$$c_{mn} dm_i + c_{nn} dn_i = \left[\tau^{1-\sigma} \tilde{A}_{i,a_i}(s_{F,i}) + \tilde{A}_{i,a_i}(s_i) \right] da_i,$$

where

$$\tilde{A}_{i,a_i} = \frac{\partial A_i}{\partial a_i} + A'_i \frac{\partial s_i}{\partial a_i}.$$

Solving these equations we obtain

$$\frac{dm_i}{da_i} \frac{D}{1 - \tau^{1-\sigma}} = [(r + \theta) \psi'_{n,i}(n_i) - A'_i(s_i) s'_i(n_i)] \tilde{A}_{i,a_i}(s_{F,i}) + \tau^{1-\sigma} A'_i(s_{F,i}) s'_i(n_{F,i}) \tilde{A}_{i,a_i}(s_i),$$

$$\frac{dn_i}{da_i} D = (r + \theta) \psi'_{m,i}(m_i) \left[\tau^{1-\sigma} \tilde{A}_{i,a_i}(s_{F,i}) + \tilde{A}_{i,a_i}(s_i) \right] - (1 - \tau^{1-\sigma})^2 A'_i(s_{F,i}) s'_i(n_{F,i}) \tilde{A}_{i,a_i}(s_i),$$

where $D = c_{mm}c_{nn} - c_{mn}^2 > 0$. It follows that if $\tilde{A}_{i,a_i}(s_{F,i}) > 0$ and $\tilde{A}_{i,a_i}(s_i) > 0$ then n_i is increasing in a_i and therefore a productivity improvement reduces the steady state product span while if $\tilde{A}_{i,a_i}(s_{F,i}) < 0$ and $\tilde{A}_{i,a_i}(s_i) < 0$ then n_i is declining in a_i and therefore a productivity improvement increases the steady state product span. In general, when $\tilde{A}_{i,a_i}(s_{F,i})$ and $\tilde{A}_{i,a_i}(s_i)$ have the same signs, m_i may increase or decline when a_i rises, depending on their relative size.

Nevertheless, this solution implies

$$\begin{aligned} \frac{dn_{F,i}}{da_i} D &= (r + \theta) \psi'_{m,i}(m_i) \left[\tau^{1-\sigma} \tilde{\Lambda}_{i,a_i}(s_{F,i}) + \tilde{\Lambda}_{i,a_i}(s_i) \right] \tau^{1-\sigma} \\ &\quad + \left[(r + \theta) \psi'_{n,i}(n_i) - \Lambda'_i(s_i) s'_i(n_i) \right] \tilde{\Lambda}_{i,a_i}(s_{F,i}) (1 - \tau^{1-\sigma})^2. \end{aligned}$$

Therefore the effective number of foreign products rises with a_i when $\tilde{\Lambda}_{i,a_i}(s_{F,i}) > 0$ and $\tilde{\Lambda}_{i,a_i}(s_i) > 0$ and declines with a_i when $\tilde{\Lambda}_{i,a_i}(s_{F,i}) < 0$ and $\tilde{\Lambda}_{i,a_i}(s_i) < 0$.

Next note from (30) and (31) that

$$\begin{aligned} (r + \theta) \psi'_{m,i}(n_i) dm_i &= (1 - \tau^{1-\sigma}) d\Lambda_i(s_{F,i}), \\ &= (1 - \tau^{1-\sigma}) \frac{\partial \Lambda_i(s_{F,i})}{\partial a_i} da_i + (1 - \tau^{1-\sigma}) \Lambda'_i(s_{F,i}) ds_{F,i} \end{aligned}$$

$$\begin{aligned} (r + \theta) \psi'_{n,i}(n_i) dn_i &= d\Lambda_i(s_i) + \tau^{1-\sigma} d\Lambda_i(s_{F,i}) \\ &= \frac{\partial \Lambda_i(s_i)}{\partial a_i} da_i + \Lambda'_i(s_i) ds_i + \tau^{1-\sigma} \frac{\partial \Lambda_i(s_{F,i})}{\partial a_i} da_i + \tau^{1-\sigma} \Lambda'_i(s_{F,i}) ds_{F,i}. \end{aligned}$$

Therefore in the case in which an improvement in labor productivity reduces both n_i and $n_{F,i}$, i.e., $s_{F,i} > s_c$, this implies that $\Lambda'_i(s_i) ds_i + \Lambda'_i(s_{F,i}) ds_{F,i} < 0$. Therefore either $ds_i > 0$ or $ds_{F,i} > 0$. In other words, the market share has to increase in at least one country. Moreover, if m_i declines, then $ds_{F,i} > 0$, implying that the market share rises in the foreign country.